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Crossing Numbers and Cutwidths

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Abstract

The crossing number of a graph $G = (V, E)$, denoted by $\text{cr}(G)$, is the smallest number of edge crossings in any drawing of G in the plane. We assume that the drawing is good, i.e., incident edges do not cross, two edges cross at most once and at most two edges cross in a point of the plane. Leighton [13] proved that for any n -vertex graph G of bounded degree, its crossing number satisfies $\text{cr}(G) + n = \Omega(\text{bw}^2(G))$, where $\text{bw}(G)$ is the bisection width of G . The lower bound method was extended for graphs of arbitrary vertex degrees to $\text{cr}(G) + \frac{1}{16} \sum_{v \in G} d_v^2 = \Omega(\text{bw}^2(G))$ in [16, 20], where d_v is the degree of any vertex v . We improve this bound by showing that the bisection width can be replaced by a larger parameter - the cutwidth of the graph. Our result also yields an upper bound for the path-width of G in terms of its crossing number.

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1 Introduction

The crossing number of a graph $G = (V, E)$, denoted by $\text{cr}(G)$, is the smallest number of edge crossings in any drawing of G in the plane. It represents a fundamental measure of non-planarity of graphs but is attractive from practical point of view too. It is known that the aesthetics and readability of graph-like structures (information diagrams, class hierarchies, flowcharts...) heavily depends on the number of crossings [4, 17], when the structures are visualized on a 2-dimensional medium. Another natural appearance of the problem is in the design of printed circuit boards and VLSI circuits [13]. The area of a VLSI circuit is strongly related to the crossing number of the underlying graph. The problem is NP-hard [6] and the best theoretical exact and approximation algorithms are in [5, 9]. A survey on heuristics is in [3]. Concerning crossing numbers of standard graphs, there are only a few infinite classes of graphs for which exact or tight bounds are known [12]. The main problem is the lack of efficient lower bound methods for estimating the crossing numbers of explicitly given graphs. The survey on known methods is in [19]. One of the powerful methods is based on the bisection width concept. The bisection width of a graph G is the minimum number of edges whose removal divides G into two parts having at most $2|V|/3$ vertices each. Leighton [13] proved that in any n -vertex graph G of bounded degree, the crossing number satisfies $\text{cr}(G) + n = \Omega(\text{bw}^2(G))$. The lower bound was extended to

$$\text{cr}(G) + \frac{1}{16} \sum_{v \in V} d_v^2 \geq \frac{1}{40} \text{bw}^2(G)$$

in [16, 20], where d_v is the degree of any vertex v . We improve this bound by showing that the bisection width can be replaced by a larger parameter - the cutwidth of the graph, denoted by $\text{cw}(G)$ and defined as follows. Let $G = (V, E)$ be a graph. Let $\phi : V \rightarrow \{1, 2, 3, \dots, |V|\}$ be an injection. Then

$$\text{cw}(G) = \min_{\phi} \max_i |\{uv \in E : \phi(u) \leq i < \phi(v)\}|.$$

Note that the cutwidth is a standard graph invariant appearing e.g. in the linear VLSI layouts [21], and is related to such a classical topic like the discrete isoperimetric problem [1]. We prove

$$\text{cr}(G) + \frac{1}{16} \sum_{v \in G} d_v^2 \geq \frac{1}{1176} \text{cw}^2(G). \quad (1)$$

Ignoring the constant factors, the improvement is evident as $\text{cw}(G) \geq \text{bw}(G)$ and there are connected graphs with $\text{bw}(G) = 1$ but with arbitrarily large cutwidth. For example, let G be a graph obtained by joining two $K_{\frac{n}{2}}$'s by an edge. Then clearly $\text{cw}(G) = \Omega(n^2)$. If $\text{cw}(G) \approx \text{bw}(G)$ then the bisection lower bound is better up to a constant factor because of the small constant in our estimation. An improvement on it remains an open problem. Anyway, the

aim of this note is to show that from the asymptotic point of view, the graph invariant that essentially influences the crossing number is not the bisection width but the cutwidth. The new crossing number lower bound is tight up to a constant factor for a large class of graphs. Following Pach and Tóth [15], for almost all n -vertex and m -edge graphs G , $\text{bw}(G) \geq m/10$, where $m \geq 10n$. As $\text{cw}(G) \geq \text{bw}(G)$, the lower bound (1) implies $\text{cr}(G) = \Omega(\text{cw}^2(G)) = \Omega(m^2)$. On the other hand, trivially $\text{cr}(G) = O(m^2) = O(\text{cw}^2(G))$.

Moreover, the additive term $\sum_{v \in G} d_v^2$ cannot be removed, since the crossing number of any planar graph is 0 and there exists a planar graph P (e.g. the star) such that $\text{cw}^2(P) = \Omega(\sum_{v \in P} d_v^2)$.

As a byproduct we obtain the following contribution to topological graph theory. The path-decomposition of a graph G is a sequence $D = X_1, X_2, \dots, X_r$ of vertex subsets of G , such that every edge of G has both ends in some set X_i and if a vertex of G occurs in some sets X_i and X_j with $i < j$, then the same vertex occurs in all sets X_k with $i < k < j$. The width of D is the maximum number of vertices in any X_i minus 1. The path-width of G , $\text{pw}(G)$, is the minimum width over all path-decompositions of G .

A graph $G = (V, E)$ is k -crossing critical if $\text{cr}(G) = k$ and $\text{cr}(G - e) < \text{cr}(G)$, for all edges $e \in E$. Hliněný [10] proved that $\text{pw}(G) \leq 2^{f(k)}$, where $f(k) = O(k^3 \log k)$. This answers an open question of Geelen et al. [8] whether crossing critical graphs with bounded crossing numbers have bounded path-widths.

Our result implies another relation between pathwidths and crossing numbers. If $\text{cr}(G) = k$, then $\text{pw}(G) \leq 9\sqrt{k} + \sum_{v \in V} d_v^2$, without the crossing-criticality assumption.

2 A New Lower Bound

We will make use of the following theorem [7].

Theorem 1 *Let $G = (V, E)$ be a planar graph with non-negative weights on its vertices that sum up to one and every weight is at most $\frac{2}{3}$. Let d_v denote the degree of any vertex v . Then there exists at most $\frac{\sqrt{3} + \sqrt{2}}{2} \sqrt{\sum_{v \in V} d_v^2}$ edges whose removal divides G into disjoint subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that the weight of each is at most $\frac{2}{3}$.*

Theorem 2 implies an upper bound for the cutwidth of planar graphs which deserves an independent interest.

Theorem 2 *For any planar graph $G = (V, E)$*

$$\text{cw}(G) \leq \frac{6\sqrt{2} + 5\sqrt{3}}{2} \sqrt{\sum_{v \in V} d_v^2},$$

where d_v is the degree of any vertex v .

Proof: Apply Theorem 1 to G . Assign weights to vertices:

$$\text{weight}(u) = \frac{d_u^2}{\sum_{v \in V} d_v^2}.$$

1. Let $\text{weight}(u) \leq \frac{2}{3}$ for all u . By deleting $\frac{\sqrt{3}+\sqrt{2}}{2} \sqrt{\sum_{v \in V} d_v^2}$ edges we get graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that for $i = 1, 2$ $\text{weight}(V_i) \leq \frac{2}{3}$, which implies

$$\sum_{v \in V_i} d_v^2 \leq \frac{2}{3} \sum_{v \in V} d_v^2.$$

2. Assume there exists a vertex u such that $\text{weight}(u) > \frac{2}{3}$. By deleting edges adjacent to u we get disjoint subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where G_2 is a one vertex graph. We have $\text{weight}(V_1) = 1 - \text{weight}(u) < \frac{2}{3}$ and

$$\sum_{v \in V_1} d_v^2 < \frac{2}{3} \sum_{v \in V} d_v^2.$$

The number of edges between G_1 and G_2 is

$$d_u \leq \frac{\sqrt{3} + \sqrt{2}}{2} \sqrt{\sum_{v \in V} d_v^2}.$$

Placing the graphs G_1 and G_2 consecutively on the line and adding the deleted edges we obtain the estimation

$$\text{cw}(G) \leq \max\{\text{cw}(G_1), \text{cw}(G_2)\} + \frac{(\sqrt{3} + \sqrt{2})}{2} \sqrt{\sum_{v \in V} d_v^2}.$$

Solving the recurrence we find

$$\text{cw}(G) \leq \frac{\sqrt{3} + \sqrt{2}}{2} \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{i/2} \sqrt{\sum_{v \in V} d_v^2} = \frac{6\sqrt{2} + 5\sqrt{3}}{2} \sqrt{\sum_{v \in V} d_v^2}.$$

□

Our main result is

Theorem 3 *Let $G = (V, E)$ be a graph. Let d_v denote the degree of any vertex v . Then the crossing number of G satisfies*

$$\text{cr}(G) + \frac{1}{16} \sum_{v \in V} d_v^2 \geq \frac{1}{1176} \text{cw}^2(G).$$

Proof: Consider a drawing of G with $\text{cr}(G)$ crossings. Introducing a new vertex at each crossing results in a plane graph H with $\text{cr}(G) + n$ vertices. By Theorem 2 we have

$$\text{cw}(H) \leq \frac{6\sqrt{2} + 5\sqrt{3}}{2} \sqrt{\sum_{v \in H} d_v^2} = \frac{6\sqrt{2} + 5\sqrt{3}}{2} \sqrt{\sum_{v \in G} d_v^2 + 16\text{cr}(G)}.$$

Finally, note that $\text{cw}(G) \leq \text{cw}(H)$, which proves the claim. \square

This result immediately gives an upper bound for the path-width of G in terms of its crossing number as the result of Kinnersley [11] implies that $\text{pw}(G) \leq \text{cw}(G)$.

Corollary 1 *Let $G = (V, E)$ be a graph. Then*

$$\text{pw}(G) < \frac{6\sqrt{2} + 5\sqrt{3}}{2} \sqrt{16\text{cr}(G) + \sum_{v \in V} d_v^2}.$$

3 Final Remarks

We proved a new lower bound formula for estimating the crossing numbers of graphs. The former method was based on the bisection width of graphs. Our method replaces the bisection width by a stronger parameter - the cutwidth. While the bisection width of a connected graph can be just one edge, which implies a trivial lower bound only, the cutwidth based method gives nontrivial lower bounds in most cases. A drawback of the method is the big constant factor in the formula.

A natural question arises how to find or estimate the cutwidth of a graph. The most frequent approach so far was its estimation from below by the bisection width. This of course degrades the cutwidth method to the bisection method. Provided that $\text{cw}(H)$ is known or estimated from below, for some graph H , we can use a well-known relation $\text{cw}(G) \geq \text{cw}(H)/\text{cg}(H, G)$, see [18], where is the congestion of G in H defined as follows. Consider an injective mapping of vertices of H into the vertices of G and a mapping of edges of H into paths in G . Take the maximal number of paths traversing an edge. Minimizing this maximum over all possible mappings gives $\text{cg}(H, G)$. Another possibility is to use the strong relation of the cutwidth problem to the so called discrete edge isoperimetric problem [1]. Informally, the problem is to find, for a given k , a k -vertex subset of a graph with the smallest “edge boundary”. A good solution to the isoperimetric problem provides a good lower bound for the cutwidth.

Recently Pach and Tardos [14] proved another relation between crossing numbers and a special edge cut of a graph (Corollary 5), which resembles our Theorem 2.3. But neither of the two statements implies the other.

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