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Drawing Subcubic 1-Planar Graphs with Few Bends, Few Slopes, and Large Angles

Philipp Kindermann[1] *Fabrizio Montecchiani*[2] *Lena Schlipf*[3] *André Schulz*[4]

[1]Universität Würzburg, Germany
[2]Università degli Studi di Perugia, Italy
[3]Universität Tübingen, Germany
[4]FernUniversität in Hagen, Germany

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Abstract. We show that the 1-planar slope number of 3-connected cubic 1-planar graphs is at most four when edges are drawn as polygonal curves with at most one bend each, that is, any such graph admits a drawing with at most one bend per edge and such that the number of distinct slopes used by the edge segments is at most four. This bound is obtained by drawings whose angular and crossing resolution is at least $\pi/4$. On the other hand, if the embedding is fixed, then there is a 3-connected cubic 1-planar graph that needs three slopes when drawn with at most one bend per edge. We also show that two slopes always suffice for 1-planar drawings of subcubic 1-planar graphs with at most two bends per edge. This bound is obtained with angular resolution $\pi/2$ and the drawing has crossing resolution $\pi/2$ (i.e., it is a RAC drawing). Finally, we prove lower bounds for the slope number of straight-line 1-planar drawings in terms of number of vertices and maximum degree.

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E-mail addresses: philipp.kindermann@uni-wuerzburg.de (Philipp Kindermann) fabrizio.montecchiani@unipg.it (Fabrizio Montecchiani) lena.schlipf@fernuni-hagen.de (Lena Schlipf) andre.schulz@fernuni-hagen.de (André Schulz)

1 Introduction

A graph is 1-*planar* if it can be drawn in the plane such that each edge is crossed at most once. The notion of 1-planarity naturally extends planarity and received considerable attention since its first introduction by Ringel in 1965 [38], as witnessed by recent surveys [6, 17, 32]. Despite the efforts made in the study of 1-planar graphs, only few results are known concerning their geometric representations (see, e.g., [1, 4, 8, 10, 11, 14]). In this paper, we study the existence of 1-planar drawings that simultaneously satisfy the following properties: edges are polylines using few bends and few distinct slopes for their segments, edge crossings occur at large angles, and pairs of edges incident to the same vertex form large angles. For example, Figure 3d shows a 1-bend drawing of a 1-planar graph (i.e., a drawing in which each edge is a polyline with at most one bend) using 4 distinct slopes, such that edge crossings form angles at least $\pi/4$, and the angles formed by edges incident to the same vertex are at least $\pi/4$. In what follows, we recall some literature concerning the problems of computing polyline drawings with few bends and few slopes or with few bends and large angles.

Related work. The *k-bend (planar) slope number* of a (planar) graph G with maximum vertex degree Δ is the minimum number of distinct edge slopes needed to obtain a (planar) drawing of G such that each edge is a polyline with at most k bends. When $k = 0$, this parameter is simply known as the *(planar) slope number* of G. Clearly, if G has maximum vertex degree Δ, at least $\lceil \Delta/2 \rceil$ slopes are needed for any k. While there exist non-planar graphs with $\Delta \geq 5$ whose slope number is unbounded with respect to Δ [3, 37], Keszegh et al. [28] proved that the planar slope number is bounded by $2^{O(\Delta)}$. Several authors improved this bound for subfamilies of planar graphs (see, e.g., [25, 31, 33]).

Concerning k-bend drawings, Angelini et al. [2] proved that the 1-bend planar slope number is at most $\Delta - 1$ (for $\Delta \geq 4$), while $\frac{3}{4}(\Delta - 1)$ slopes may be needed as shown by Knauer and Walczak [30]. Keszegh et al. [28] proved that the 2-bend planar slope number is $\lceil \Delta/2 \rceil$ (which matches the trivial lower bound).

Special attention has been paid in the literature to the slope number of *(sub)cubic* graphs, i.e., graphs having vertex degree (at most) three. Mukkamala and Pálvölgyi [36] showed that the four slopes $\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$ suffice for every cubic graph. For planar graphs, Kant [26] and independently Dujmović et al. [18] proved that cubic 3-connected planar graphs have planar slope number three disregarding the slopes of three edges on the outer face, while Di Giacomo et al. [16] proved that the planar slope number of (not necessarily 3-connected) subcubic planar graphs is at most four, which is worst-case optimal.

We note that the slope number problem is related to orthogonal drawings, which are planar and with slopes $\{0, \frac{\pi}{2}\}$ [19], and with octilinear drawings, which are planar and with slopes $\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$ [5]. All planar graphs with maximum vertex degree $\Delta \leq 4$ (except the graph of the octahedron) admit 2-bend orthogonal drawings [7, 34], and planar graphs have octilinear drawings with no bends if $\Delta \leq 3$ [16, 26], with one bend if $\Delta \leq 5$ [5], and with two bends if $\Delta \leq 8$ [28].

Of particular interest for us is the *k-bend* 1-*planar slope number* of 1-planar graphs, i.e., the minimum number of distinct edge slopes needed to compute a 1-planar drawing of a 1-planar graph such that each edge is a polyline with at most $k \geq 0$ bends (we recall that not all 1-plane graphs have a straight-line drawing [24]). For this problem, Di Giacomo et al. [15] proved an $O(\Delta)$ upper bound for the 1-planar slope number ($k = 0$) of outer 1-planar graphs, i.e., graphs having a 1-planar drawing with all vertices on the external boundary.

Finally, the *angular resolution* and the *crossing resolution* of a drawing are defined as the minimum angle between two consecutive segments incident to the same vertex or crossing, respectively (see, e.g., [20, 23, 35]). A drawing is *RAC* if its crossing resolution is $\pi/2$. Eades and Liotta proved that 1-planar graphs may not have straight-line RAC drawings [21], while Bekos et al. [4] proved that every 1-planar graph has a 1-bend RAC drawing, but their algorithm might require exponential area. Recently, Chaplick et al. [11] modified the algorithm of Bekos et al. such that a given embedding is preserved.

Our contribution. We prove upper and lower bounds on the k-bend 1-planar slope number of 1-planar graphs, when $k \in \{0, 1, 2\}$. Our results are based on techniques that lead to drawings with large vertex and crossing resolution. More precisely, our results can be summarized as follows.

(i) We prove that every 3-connected cubic 1-planar graph admits a 1-bend 1-planar drawing that uses at most four distinct slopes and has both angular and crossing resolution $\pi/4$.

(ii) We prove that every subcubic 1-planar graph admits a 2-bend 1-planar drawing that uses at most two distinct slopes and has both vertex and crossing resolution $\pi/2$. These bounds on the number of slopes and on the angular/crossing resolution are clearly worst-case optimal.

(iii) We exhibit a 3-connected cubic 1-plane graph for which any embedding-preserving 1-bend drawing uses at least three distinct slopes. The lower bound holds even if we are allowed to change the outer face.

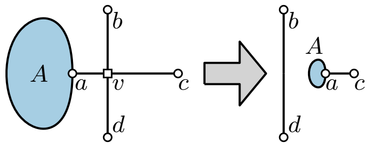
(iv) We present 2-connected subcubic 1-plane graphs with n vertices such that any embedding-preserving straight-line drawing uses $\Omega(n)$ distinct slopes, and

(v) We show 3-connected 1-plane graphs with maximum degree $\Delta \geq 3$ such that any embedding-preserving straight-line drawing uses at least $9(\Delta - 1)$ distinct slopes, which implies that at least 18 slopes are needed if $\Delta = 3$.

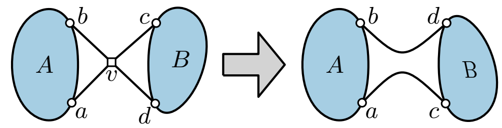
Paper organization. Section 2 contains preliminaries and basic results that will be used throughout the paper. The 1-bend 1-planar slope number is studied in Section 3, while the 2-bend 1-planar slope number is studied in Section 4. The lower bounds for the k-bend 1-planar slope number are in Section 5.1 ($k = 1$) and in Section 5.2 ($k = 0$). We conclude in Section 6 with a list of interesting open problems that arise from our research.

2 Preliminaries and Basic Results

Drawings and embeddings. We only consider *simple* graphs with neither self-loops nor multiple edges. A *drawing* Γ of a graph G maps each vertex of G to a point of the plane and each edge to a simple open Jordan arc between its endpoints. We always refer to *simple* drawings where two edges can share at most one point, which is either a common endpoint or a proper intersection. A drawing divides the plane into topologically connected regions, called *faces*; the unbounded region is called the *outer face*. For a planar (i.e., crossing-free) drawing, the boundary of a face consists of vertices and edges, while for a non-planar drawing the boundary of a face may also contain crossings and parts of edges. An *embedding* of a graph G is an equivalence class of drawings of G that define the same set of faces and the same outer face. A *plane graph* is a graph with a fixed planar embedding. Similarly, a 1-*plane graph* is a graph with a fixed 1-planar embedding. Given a 1-plane graph G, the *planarization* G^* of G is the plane graph obtained by replacing each crossing of G with a *dummy vertex*. To avoid confusion, the vertices of G^* that are not dummy vertices are called *real vertices*. Moreover, we call the edges of G^* that are incident to a dummy vertex



(a) (v, a) is a bridge



(b) v has no bridge

Figure 1: Eliminating dummy cutvertices from G^*

fragments. The next lemma will be used in the following and can be of independent interest, as it extends a similar result by Fabrici and Madaras [22].

Lemma 1 *Let $G = (V, E)$ be a 1-plane graph and let G^* be its planarization. We can re-embed G such that each edge is still crossed at most once and (i) no cutvertex of its planarization is a dummy vertex, and (ii) if G is 3-connected, then its planarization is 3-connected.*

Proof: We first show how to iteratively remove all cutvertices from G^* that are dummy vertices. Suppose that there is a cutvertex v in G^* that is a dummy vertex. Let a, b, c, d be the neighbors of v in clockwise order, so the edges (a, c) and (b, d) cross in G.

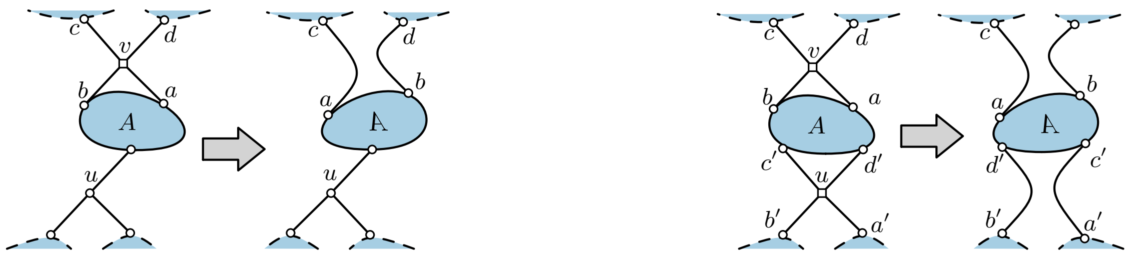
First, assume that one of the four edges of v, say (v, a), is a bridge, so removing v from G^* gives a connected component A that contains neither b, c, d; see Figure 1a. We shrink A and move it along the edge (a, c) such that the crossing between (a, c) and (b, d) is eliminated from G, and no new crossing is introduced.

We can now assume that there is no bridge at v, so removal of v divides G^* into two components A and B. If a and c lie in the same component, G is disconnected as there is no path from a to b; hence, assume w.l.o.g. that $a, b \in A$ and $c, d \in B$; see Figure 1b. Hence, there is a simple closed curve that passes through v and separates A and B. The existence of this curve ensures that we can modify A and B independently. In particular, we can transform B as follows: we shrink it, mirror it on any axis, and possibly rotate it such that, afterwards, we can reroute the edge (a, c) along (a, v) and (v, c) and reroute (b, d) along (b, v) and (v, d). This eliminates the crossing between (a, c) and (b, d) from G and again it creates no further crossings.

This shows the first part of the lemma. For the second part, suppose that G is 3-connected and G^* has no dummy vertex as a cutvertex; otherwise, apply the first part of the lemma. Assume that there is a separation pair u, v in G^* where v is a dummy vertex. Let again a, b, c, d be the neighbors of v in clockwise order.

First, assume that u is a real vertex. The removal of u and v splits G^* in at most four connected components. If one of these connected components contains exactly one neighbor of v, say a, there are at most two vertex-disjoint paths from a to c in G: the edge (a, c) and one path via u. But this contradicts 3-connectivity. Hence, there are two connected components A and B that contain two neighbors of v each; assume w.l.o.g. that A contains a. If A contains a and c, u is a cutvertex in G, which contradicts 3-connectivity. If A contains a and b, we redraw A as described in the previous case, reroute the edge (b, d) along (b, v) and (v, d) and reroute (a, c) along (a, v) and (v, c); see Figure 2a. This eliminates the crossing between (a, c) and (b, d) from G. If A contains a and d, we proceed analogously.

Second, assume that u is also a dummy vertex with neighbors a', b', c', d' in clockwise order. Removal of u and v splits G^* in at most four connected components. If one of these connected components contains exactly one neighbor of v, say a, and exactly one neighbor of u, say a', then



(a) A contains a, b and u is a real vertex.

(b) A contains a, b and u is a dummy vertex.

Figure 2: Eliminating a dummy separation pair from G^*.

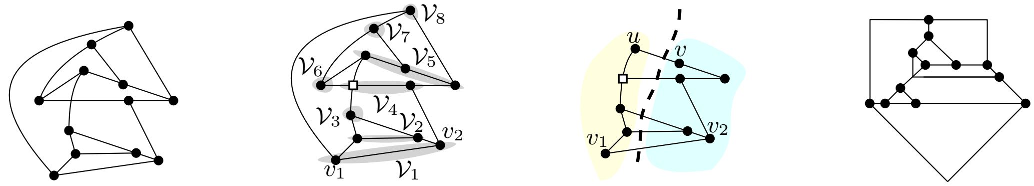
there are at most two vertex-disjoint paths from a to c in G: the edge (a, c) and one path via the edge (a', c'). But this contradicts 3-connectivity. If one of the connected components contains exactly three neighbors of v, say a, b, c, and exactly one neighbor of u, say a', then d, a' is a separation pair in G, as it separates a, b, c from b', c', d'. Again, this contradicts 3-connectivity. Hence, each of these connected components contains exactly two neighbors of one of v and u. Let A be a connected component and assume w.l.o.g. that it contains a and one more neighbor of v. If A contains a and c, there is some neighbor of u that is not in A, say a'. Since $(a, c) \in A$ and $b, d \notin A$, all paths from a to a' in G have to traverse the edge (a', c') or (b', d'), so there are at most two of them; a contradiction to 3-connectivity. If A contains a and b, we flip A and reroute the edges (a, c) and (b, d) to eliminate their crossing from G. Note that, if A contains also exactly two neighbors of u, this also eliminates the crossing between (a', c') and (b', d'); see Figure 2b. If A contains a and d, we proceed analogously.

Each step reduces the number of crossings in the embedding, so it terminates with an embedding that has a 3-connected planarization. □

Slopes and ports. A drawing Γ is *straight-line* if all its edges are mapped to segments, or it is *k-bend* if each edge is mapped to a chain of segments with at most $k > 0$ bends. The *slope* of an edge segment of Γ is the slope of the line containing this segment. For convenience, we measure the slopes by their angle with respect to the x-axis. Let $\mathcal{S} = \{\alpha_1, \dots, \alpha_t\}$ be a set of t distinct slopes. The *slope number* of a k-bend drawing Γ is the number of distinct slopes used for the edge segments of Γ. An edge segment of Γ uses the *north (N) port* (*south (S) port*) of a vertex v if it has slope $\pi/2$ and v is its bottommost (topmost) endpoint. We can define analogously the *west (W)* and *east (E)* ports with respect to the slope 0, the *north-west (NW)* and *south-east (SE)* ports with respect to slope $3\pi/4$, and the *south-west (SW)* and *north-east (NE)* ports with respect to slope $\pi/4$. Any such port is *free* for v if there is no edge that attaches to v by using it.

Orderings. We will use a decomposition technique called *canonical ordering* [27]. Let $G = (V, E)$ be a 3-connected plane graph. Let $\delta = \{\mathcal{V}_1, \dots, \mathcal{V}_K\}$ be an ordered partition of V, that is, $\mathcal{V}_1 \cup \dots \cup \mathcal{V}_K = V$ and $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ for $i \neq j$. Let G_i be the subgraph of G induced by $\mathcal{V}_1 \cup \dots \cup \mathcal{V}_i$ and denote by C_i the outer face of G_i. The partition δ is a canonical ordering of G if:

1. $\mathcal{V}_1 = \{v_1, v_2\}$, where v_1 and v_2 lie on the outer face of G and $(v_1, v_2) \in E$.
2. $\mathcal{V}_K = \{v_n\}$, where v_n lies on the outer face of G, $(v_1, v_n) \in E$.
3. Each C_i $(i > 1)$ is a cycle containing (v_1, v_2).



(a) G (b) δ (c) uv-cut (d)

Figure 3: (a) A 3-connected cubic 1-plane graph G. (b) A canonical ordering δ of the planarization G^* of G; the real (dummy) vertices are black points (white squares). (c) The edges crossed by the dashed line are a uv-cut of G_5 with respect to (u, v); the two components have a yellow and a blue background, respectively. (d) A 1-bend 1-planar drawing with 4 slopes of G.

4. Each G_i is 2-connected and internally 3-connected, that is, removing any two interior vertices of G_i does not disconnect it.
5. For each $i \in \{2, \ldots, K-1\}$, one of the following conditions holds:
   (a) $\mathcal{V}_i$ is a *singleton* v^i that lies on C_i and has at least one neighbor in $G \setminus G_i$;
   (b) $\mathcal{V}_i$ is a *chain* $\{v_1^i, \ldots, v_l^i\}$, both v_1^i and v_l^i have exactly one neighbor each in C_{i-1}, and $v_2^i, \ldots, v_{l-1}^i$ have no neighbor in C_{i-1}. Since G is 3-connected, each v_j^i has at least one neighbor in $G \setminus G_i$.

Let v be a vertex in $\mathcal{V}_i$, then its neighbors in G_{i-1} (if G_{i-1} exists) are called the *predecessors* of v, while its neighbors in $G \setminus G_i$ (if G_{i+1} exists) are called the *successors* of v. In particular, every singleton has at least two predecessors and at least one successor, while every vertex in a chain has either zero or one predecessor and at least one successor. Kant [27] proved that a canonical ordering of G always exists and can be computed in $O(n)$ time; the technique in [27] is such that one can arbitrarily choose two adjacent vertices u and w on the outer face so that $u = v_1$ and $w = v_2$ in the computed canonical ordering.

An n-vertex *planar st-graph* $G = (V, E)$ is a plane acyclic directed graph with a single source s and a single sink t, both on the outer face [13]. An *st-ordering* of G is a numbering $\sigma : V \to \{1, 2, \ldots, n\}$ such that for each edge $(u, v) \in E$, it holds $\sigma(u) < \sigma(v)$ (thus $\sigma(s) = 1$ and $\sigma(t) = n$). For an st-graph, an st-ordering can be computed in $O(n)$ time (see, e.g., [12]) and every biconnected undirected graph can be oriented to become a planar st-graph (also in linear time).

3 1-bend Drawings of 3-connected Cubic 1-planar Graphs

In this section we prove the following result.

Theorem 1 *Every 3-connected cubic 1-planar graph admits a 1-bend 1-planar drawing using only slopes in the set* $\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$ *and having both angular and crossing resolution at least* $\frac{\pi}{4}$.

Let G be a 3-connected cubic 1-plane graph (see, e.g., Figure 3a). We begin by listing a few assumptions on G. Let G^* be the planarization of G. We can assume that G^* is 3-connected (else we can re-embed G by Lemma 1). For the description of the algorithm, we assume that there exists some edge (v_1, v_2) whose vertices are both real. Later, we describe how to adjust the algorithm if no such edge exists. We choose as outer face of G a face containing (v_1, v_2) (see Figure 3b).

Let $\delta = \{\mathcal{V}_1, \ldots, \mathcal{V}_K\}$ be a canonical ordering of G^*, let G_i be the graph obtained by adding the first i sets of δ, and let C_i be the outer face of G_i. We give some definitions and notation that

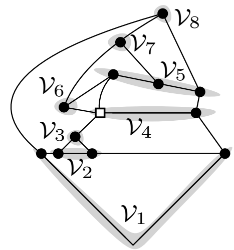


Figure 4: The left vertex of $\mathcal{V}_4$ has two L-successors, one in $\mathcal{V}_5$ and one in $\mathcal{V}_6$. The vertex is L-attachable in G_4 and in G_5.

will be useful in the following. Note that a real vertex v of G_i can have at most one successor w in some set $\mathcal{V}_j$ with $j > i$. We call w an *L-successor* (resp., *R-successor*) of v if v is the leftmost (resp., rightmost) neighbor of $\mathcal{V}_j$ on C_i. Similarly, a dummy vertex x of G_i can have at most two successors in some sets $\mathcal{V}_j$ and $\mathcal{V}_l$ with $l \geq j > i$. In both cases, a vertex v of G_i having a successor in some set $\mathcal{V}_j$ with $j > i$ is called *attachable*. We call v *L-attachable* (resp., *R-attachable*) if v is attachable and it has an L-successor (resp., R-successor) in $G \setminus G_i$. See Figure 4. The upward edge connecting v to its first L-successor (resp., first R-successor) uses either the slope 0 or $\pi/4$ (resp., 0 or $3\pi/4$). Also, if both the successors of v are L-successors (resp., R-successors), the upward edge connecting v to its second L-successor (resp., second R-successor) uses slope $\pi/2$.

Let u and v be two vertices of C_i, for $i > 1$. Denote by $P_i(u, v)$ the path of C_i having u and v as endpoints and that does not contain (v_1, v_2). Vertices u and v are *consecutive* if they are both attachable and if $P_i(u, v)$ does not contain any other attachable vertex. We say that u and v are *L-consecutive* (resp., *R-consecutive*) if they are consecutive, u lies to the left (resp., right) of v on C_i, and u is L-attachable (resp., R-attachable). Given two consecutive vertices u and v of C_i and an edge e of C_i (and in particular of $P_i(u, v)$), a *uv-cut of G_i with respect to e* is a set of edges of G_i that contains both e and (v_1, v_2) and whose removal disconnects G_i into two components, one containing u and one containing v (see Figure 3c).

We are now ready to describe our drawing technique. We aim at computing an embedding-preserving drawing Γ_i of G_i, for $i = 2, \ldots, K$, by adding one by one the sets of δ.

Definition 1 (Valid drawing.) *A drawing Γ_i of G_i is* valid*, if:*

- **P1** *It uses only slopes in the set $\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$;*
- **P2** *It is a 1-bend drawing such that the union of any two edge fragments that correspond to the same edge in G is drawn with (at most) one bend in total.*

A valid drawing Γ_K of G_K will coincide with the desired drawing of G, after replacing dummy vertices with crossing points.

Construction of Γ_2. We distinguish three cases, based on whether $\mathcal{V}_2$ is a real singleton, a dummy singleton, or a chain. In all cases, the edge between v_1 and v_2 uses the SE port of v_1 and the SW port of v_2 and, thus, has one bend. If a vertex $v \in \mathcal{V}_2$ is real, its adjacent edges use the W port and the E port. If v is dummy, we use the S port and the E port (resp., N port) of v if it has two R-successors (resp., two L-successors), and the W port and the SE port otherwise. Note that, in any case, each of these edges has a horizontal segment. The cases are illustrated in Figure 5 and it is immediate to verify that all of them represent a valid drawing of G_2.

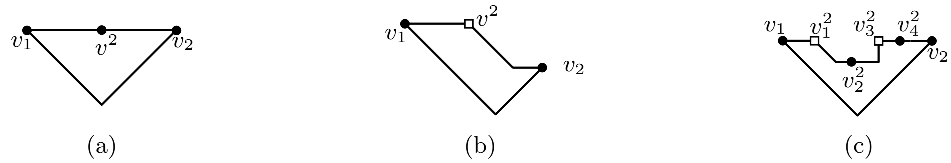


Figure 5: Construction of Γ_2: (a) $\mathcal{V}_2$ is a real singleton; (b) $\mathcal{V}_2$ is a dummy singleton; (c) $\mathcal{V}_2$ is a chain where v_3^2 is a vertex with two R-successors.

Construction of Γ_i, for $2 < i < K$. We now show how to compute a valid drawing of G_i, for $i = 3, \ldots, K-1$, by incrementally adding the sets of δ.

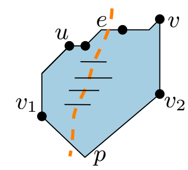


Figure 6: Stretching of Γ_i.

We aim at constructing a valid drawing Γ_i that is also stretchable, i.e., that satisfies the two properties below; see Figure 6 for an illustration. These two properties will be useful to prove Lemma 2, which defines a standard way of stretching a drawing by lengthening horizontal segments.

Definition 2 (Stretchable drawing.) *A valid drawing Γ_i of G_i is* stretchable *(see Figure 6 for an illustration), if:*

- **P3** *The edge (v_1, v_2) is drawn with two segments s_1 and s_2 that meet at a point p. Segment s_1 uses the SE port of v_1 and s_2 uses the SW port of v_2. Also, p is the lowest point of Γ_i, and no other point of Γ_i belongs to the two lines that contain s_1 and s_2.*
- **P4** *For every pair of consecutive vertices u and v of C_i with u left of v on C_i, it holds that:*
  - *(a) If u is L-attachable, then for each vertical segment s in $P_i(u,v)$, traversed upwards when going from u to v, there is a horizontal segment in the subpath before s; Symmetrically, if v is R-attachable, then for each vertical segment s in $P_i(u,v)$ there is a horizontal segment in the subpath before s, if s is traversed upwards when going from v to u;*
  - *(b) if u is L-attachable and v is R-attachable, then $P_i(u,v)$ contains at least one horizontal segment; and*
  - *(c) for every edge e of $P_i(u,v)$ that contains a horizontal segment, there exists a uv-cut of G_i with respect to e whose edges, other than (v_1, v_2), are all drawn as horizontal segments in Γ_i with the following property: there exists a y-monotone curve that crosses all the edges of the uv-cut but no other edge.*
- **P5** *The path $P_i(v_1, v_2)$ is drawn x-monotone.*

Lemma 2 *Suppose that Γ_i is valid and stretchable, and let u and v be two consecutive vertices of C_i. If u is L-attachable (resp., v is R-attachable), then it is possible to modify Γ_i such that any ray with slope $\pi/4$ (resp., $3\pi/4$) that originates at u (resp., at v) and that intersects the outer face of Γ_i does not intersect any edge segment with slope $\pi/2$ of $P_i(u,v)$. Also, the modified drawing is still valid and stretchable.*

Proof: Refer to Figure 7. Suppose there is a ray h that originates at u (the argument is analogous for v) with slope $\pi/4$ that intersects the outer face of Γ_i and, in particular, some edge segment of $P_i(u,v)$. Let s be the first edge segment of $P_i(u,v)$ that is intersected by h. By the slope of h, we have that s is drawn with slope $\pi/2$. Then, s must be traversed upwards when going from u to v in $P_i(u,v)$, and thus by **P4** there is a horizontal segment before s.

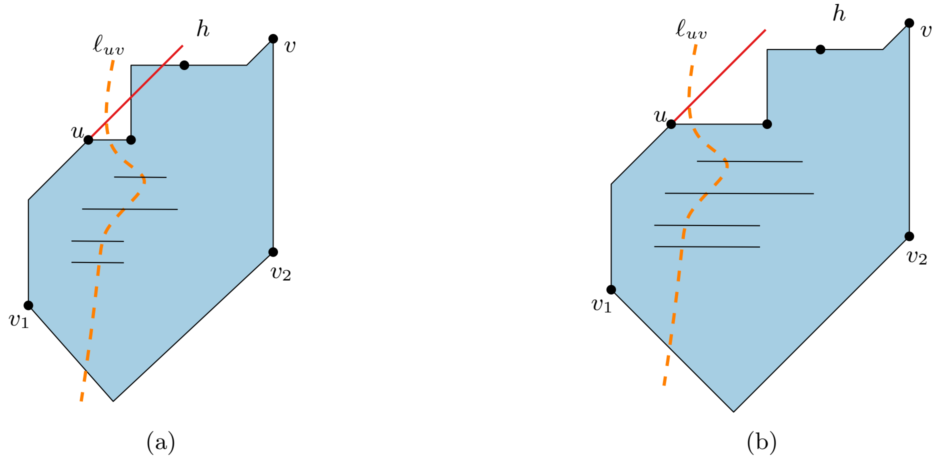


Figure 7: Illustration for Lemma 2.

Let e be the edge containing this horizontal segment. By **P4**, there is a uv-cut with respect to e and there is a y-monotone curve ℓ_{uv} that cuts the horizontal segments of this cut. Let C_u and C_v be the two components defined by the uv-cut, such that C_u contains u and C_v contains v. We shift all vertices in C_v and all edges having both end-vertices in C_v to the right by σ units, for some suitable $\sigma > 0$. All vertices in C_u and all edges having both end-vertices in C_u are not modified. Furthermore, the edges having an end-vertex in C_u and the other end-vertex in C_v are all and only the edges of the uv-cut, and thus they all contain a horizontal segment in Γ_i that can be stretched by σ units. Finally, note that (v_1, v_2) is also part of the uv-cut, but it does not contain any horizontal segment; however, by **P3** its two segments can be always redrawn by using the SE port of v_1 and the SW port of v_2. For a suitable choice of σ, this operation removes the crossing between h and s. Moreover, no new edge crossing can appear in the drawing because ℓ_{uv} intersects only the edge segments of the cut. Hence, we can repeat this procedure until all crossings between h and segments of $P_i(u, v)$ are resolved. The resulting drawing is clearly still valid and stretchable. □

Definition 3 (Attachable drawing.) *A valid and stretchable drawing Γ_i of G_i is* attachable*, if:*

P6 *At any attachable real vertex v of Γ_i, its N, NW, and NE ports are free.*

P7 *Let v be an attachable dummy vertex of Γ_i.*

- *If v has two R-successors in $G \setminus G_i$, it is drawn as in one of Cases C1s, C2, C2s, C3 in Figure 8.*
- *If v has two successors in $G \setminus G_i$ such that at most one is an R-successor, it is drawn as in one of Cases C1, C2, C2s, C3 in Figure 8.*
- *If v has only one successor which is in $G \setminus G_i$, then it is drawn as in Case C4 in Figure 8.*
- *If v has one L-successor in $G \setminus G_i$ and one successor in G_i, then it is drawn as in one of the possible cases illustrated in Figure 9, except case C1sR.*
- *If v has one R-successor in $G \setminus G_i$ and one successor in G_i, then it is drawn as in one of the possible cases illustrated in Figure 9, except case C1R.*

As a consequence of **P7**, the following observation holds.

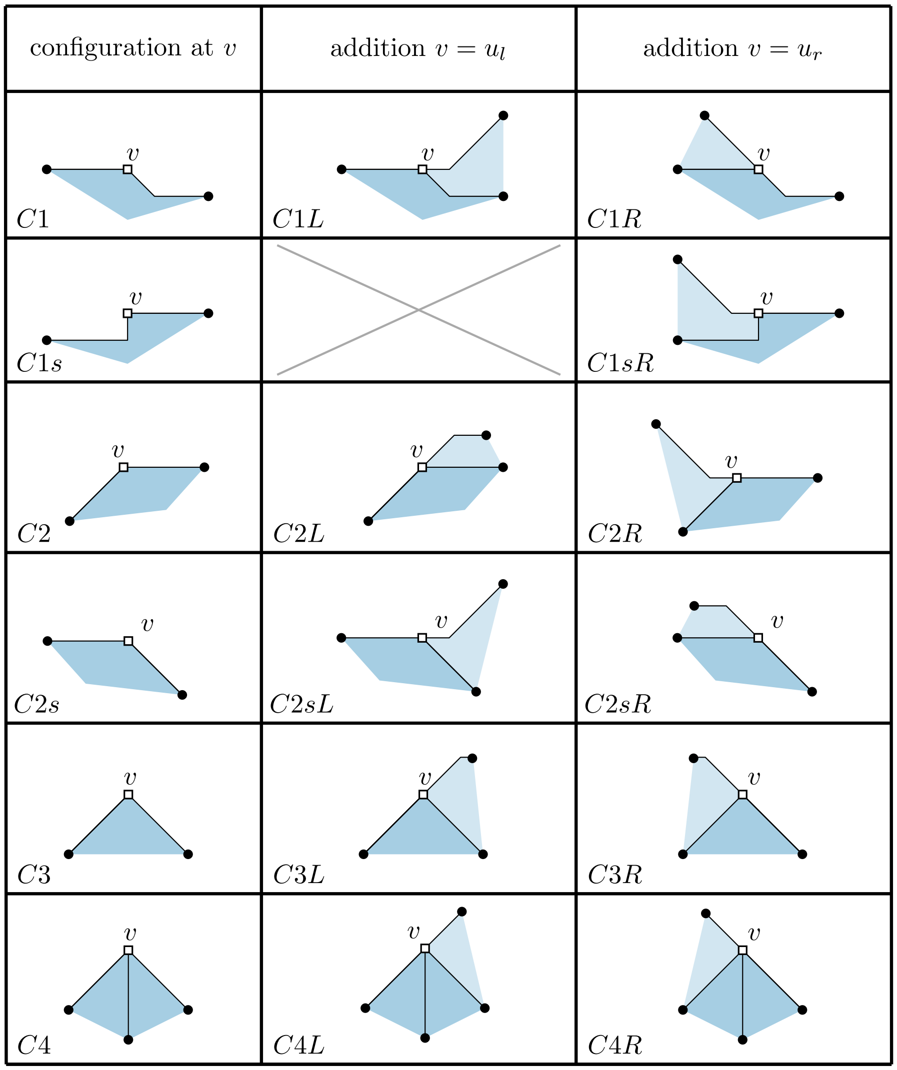


Figure 8: Cases for attaching the first success of a dummy vertex. The regions with a dark blue background are inner faces of the drawing before the addition of a singleton or of a chain; the regions with a light blue background are inner faces created by the addition of a singleton or of a chain.

Observation 1 *An edge (u, v) of an attachable drawing Γ_i of G_i is drawn as a single vertical segment if and only if v is the second L-successor or the second R-successor of u.*

Moreover, observe that Γ_2, besides being valid, is also stretchable and attachable by construction (see also Figure 5). Assume that G_{i-1} admits a valid, stretchable, and attachable drawing Γ_{i-1}, for some $2 \leq i < K-1$; we show how to add the next set $\mathcal{V}_i$ of δ so to obtain a drawing Γ_i of G_i that is valid, stretchable, and attachable. We distinguish between the following cases.

Case 1. $\mathcal{V}_i$ is a singleton, i.e., $\mathcal{V}_i = \{v^i\}$. We further distinguish based on whether v^i is real or dummy. Note that if v^i is real, it has two neighbors on C_{i-1}, while if it is dummy, it can have either two or three neighbors on C_{i-1}. Let u_l and u_r be the first and the last neighbor of v^i, respectively, when walking along C_{i-1} in clockwise direction from v_1. We will call u_l (resp., u_r) the *leftmost predecessor* (resp., *rightmost predecessor*) of v^i.

Case 1.1. Vertex v^i is real. Then, u_l and u_r are its only two neighbors in C_{i-1}. Each of u_l and u_r can be real or dummy. If u_l (resp., u_r) is real, we draw (u_l, v^i) (resp., (u_r, v^i)) with a single segment using the NE port of u_l and the SW port of v^i (resp., the NW port of u_r and the SE port of v^i).

Suppose now u_l is dummy. If u_l has just one successor or it has two successors and v^i is the first one that is placed, we distinguish between the cases of Figure 8.

If u_l has two successors and v^i is the second one that is placed, we distinguish between the various cases of Figure 9. Observe that, in order to guarantee at most one bend per edge, Case C1L requires a local reassignment of one port of u_l. Such a reassignment is always doable as the modified edge does not cross any other edge (possibly after a suitable stretching of the drawing through the edge at u_l using the W port). The edge (u_r, v^i) is handled similarly. Vertex v^i is then placed at the intersection of the lines passing through the assigned ports of u_l and u_r, which always intersect by construction.

Observe that, by construction, each port at v^i is used at most once, expect one. Namely, the S port at v^i is the only one that potentially can be used by both u_l and u_r. However, we argue that this is not the case, namely we argue that the S port at v^i is not used by both edges connecting to its predecessors. For this, we assume that both edges use the S port in order to derive a contradiction. By Observation 1, the edge (u_l, v^i) is drawn as a single vertical segment and thus uses the S port of v^i only if u_l has an L-successor x in G_{i-1}, and analogously, the edge (u_r, v^i) uses the S port of v^i only if u_r has an R-successor y in G_{i-1}; see Figure 10. Note that $x = y$ is possible but $x \neq u_r$ (otherwise, u_r would have no R-successor on C_{i-1}). Since the first edge (u_l, x) on $P_{i-1}(u_l, u_r)$ goes from a predecessor to a successor and the last edge (y, u_r) goes from a successor to a predecessor, there has to be a vertex z without a successor on the path (where $z = x$ or $z = y$ is possible). But then z has no successor in G_{i-1} and, since every vertex in a canonical order has at least one successor, z has to have a successor in $G \setminus G_{i-1}$. Hence, z is attachable. This leads to a contradiction and so u_l and u_r are not consecutive.

To avoid crossings between Γ_{i-1} and the new edges (u_l, v^i) and (u_r, v^i), we apply Lemma 2 to suitably stretch the drawing. In particular, by **P5**, possible crossings can occur only with vertical edge segments of $P_{i-1}(u_l, u_r)$.

Case 1.2. Vertex v^i is dummy. By 1-planarity, the two or three neighbors of v^i on C_{i-1} are all real. If v^i has two neighbors, we draw (u_l, v^i) and (u_r, v^i) as shown in Figure 11a, while if v^i has three neighbors, we draw (u_l, v^i) and (u_r, v^i) as shown in Figure 11b. Analogous to the previous case, vertex v^i is placed at the intersection of the lines passing through the assigned ports, which always intersect by construction, and avoiding crossings between Γ_{i-1} and the new edges (u_l, v^i) and (u_r, v^i) by applying Lemma 2. In particular, if v^i has three neighbors on C_{i-1}, say u_l, w, and

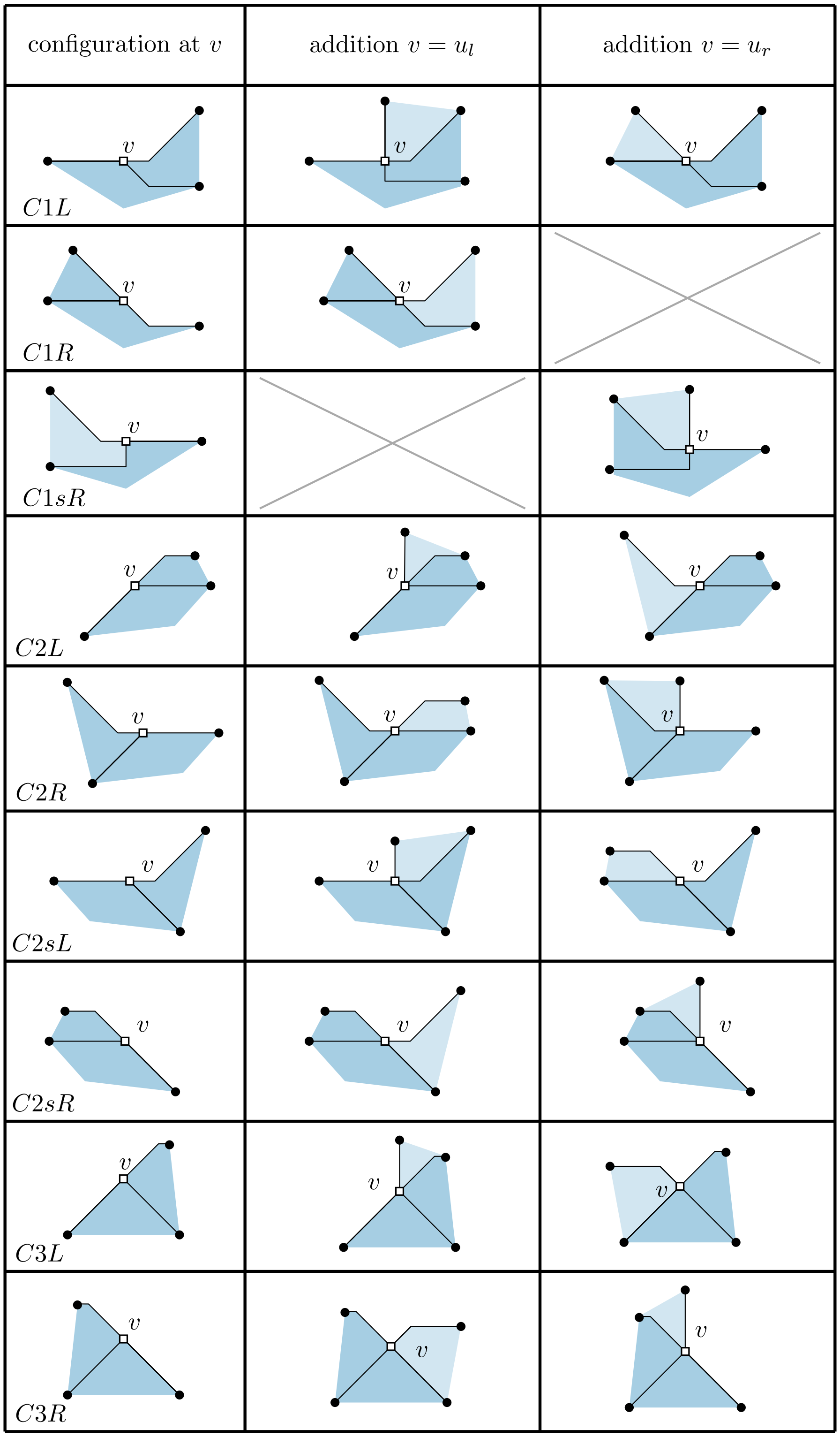


Figure 9: Cases for attaching the second successor of a dummy vertex. The regions with a dark blue background are inner faces of the drawing before the addition of a singleton or of a chain; the regions with a light blue background are inner faces created by the addition of a singleton or of a chain.

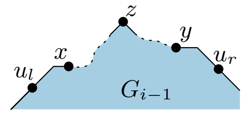


Figure 10: Illustration for Case 1.1.

(a) (b)

Figure 11: Illustration for the addition of a dummy singleton.

u_r, by **P4** there is a horizontal segment between u_l and w, as well as between w and u_r. Thus, Lemma 2 can be applied not only to resolve crossings, but also to find a suitable point where the two lines with slopes $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ meet along the line with slope $\frac{\pi}{2}$ that passes through w.

Case 2. $\mathcal{V}_i$ is a chain, i.e., $\mathcal{V}_i = \{v_1^i, v_2^i, \ldots, v_l^i\}$. We find a point as if we had to place a real vertex v whose leftmost predecessor is the leftmost predecessor of v_1^i and whose rightmost predecessor is the rightmost predecessor of v_l^i. We then draw the chain slightly below this point (i.e., in the half-plane below the horizontal line through this point) by using the same technique used to draw $\mathcal{V}_2$. Note that, if v_1^i is dummy we use case C2 of Figure 8, and similarly if v_l^i is dummy we use case C2s of Figure 8. Again, Lemma 2 can be applied to resolve possible crossings.

Lemma 3 *If a path $P_i(u, v)$ of Γ_i has a vertex whose y-coordinate is strictly smaller that those of u and v, then $P_i(u, v)$ contains a horizontal segment.*

Proof: Let w be the lowest vertex of $P_i(u, v)$. We know that the y-coordinate of w is strictly smaller than these of u and v. If w is incident to an edge of $P_i(u, v)$ that is part of a chain, then such edge contains a horizontal segment by construction. Else, since w is the lowest vertex, both its two incident edges of $P_i(u, v)$ are upward with respect to w (i.e., the vertex preceding w and the vertex following w along $P_i(u, v)$ are both successors of w), thus w has degree four and, consequently, is a dummy vertex. In this case, all possible configurations are shown in Figure 9, and in each of them there is an edge incident to w that contains a horizontal segment. □

Lemma 4 *Drawing Γ_{K-1} is valid, stretchable, and attachable.*

Proof: In all the cases used by our construction, we guaranteed the drawing to be valid and attachable. Concerning stretchability, observe that **P3** holds clearly throughout the construction. To show **P4** and **P5**, we use induction on $i \leq K - 1$ as follows.

In the base case $i = 2$, we have that Γ_2 is clearly stretchable by construction. When adding $\mathcal{V}_i$ to Γ_{i-1}, **P4** holds by induction for all pairs of vertices that are consecutive both in Γ_{i-1} and in Γ_i, because $P_{i-1}(u, v) = P_i(u, v)$. Similarly, **P5** holds by induction for all edges that appear on the boundary of both Γ_{i-1} and in Γ_i. For the edges added in Γ_i, one can easily verify that **P5** holds by looking at Figures 8, 9, and 11.

(a) v_n is dummy (b) v_n is real

Figure 12: Illustration for the addition of $\mathcal{V}_k$.

Let u_l and u_r be the two vertices on C_{i-1} used by $\mathcal{V}_i$ to attach to Γ_{i-1}. Observe that by **P4**(b) there is at least one horizontal edge e on $P_{i-1}(u_l, u_r)$, and that **P4**(c) holds for e by induction. Now for any edge e' on $P_i(u_l, u_r)$ that contains a horizontal segment, a uv-cut of G_i with respect to e' can be obtained by extending the uv-cut of G_{i-1} with respect to e so to pass through e' first. Hence, **P4**(c) holds for Γ_i.

Concerning **P4**(a–b), observe that the vertices in $\mathcal{V}_i$ are all attachable vertices. We distinguish the following cases.

Case A. $\mathcal{V}_i$ is singleton. Then, v^i is consecutive with either u_l or the attachable vertex w before u_l, and with either u_r or the attachable vertex w' after u_r.

Case A.1. v^i is consecutive to u_l or to u_r. In the former case, u_l has degree four and hence is dummy. The configurations at u_l are those shown as cases C1L, C2L, C2sL, C3L in Figure 8, and in each of them edge (u_l, v^i) contains a horizontal segment. In the latter case, u_r has degree four and hence is dummy. The configurations at u_r are those shown as cases C1R, C1sR, C2R, C2sR, C3R in Figure 8. In each of them, except C1R, edge (u_r, v^i) contains a horizontal segment. By **P7**, C1R is only used if u_r has no more R-successors, thus u_r is not R-attachable and we don't require a horizontal segment.

Case A.2 v^i is consecutive to w (a symmetric argument applies to w'). Then, observe first that **P4**(a–b) hold for $P_{i-1}(w, u_l) = P_i(w, u_l)$. Also, observe that if there is a horizontal segment in $P_i(w, u_l)$, then **P4**(a–b) hold for $P_i(w, v^i)$ (even if (u_l, v^i) contains a vertical segment and even if v^i is real).

Case A.2.1. w is real. Since w and v^i are consecutive, w is attachable, which implies it does not have successors in G_i. It follows that in the path $P_i(w, v^i)$, the first edge at w either has horizontal segment (and we are done) or it goes downward. In the latter case, by construction, the last edge of $P_i(w, v^i)$, namely (u_l, v^i), is drawn upward. It follows that $P_i(w, v^i)$ contains a vertex whose y-coordinates is strictly smaller than those of w and v^i, and the existence of a horizontal segment follows by Lemma 3.

Case A.2.2. w is dummy. Consider again the first edge of $P_i(w, v^i)$. If such edge goes downward, the same argument as in the previous case applies. Else, this edge is drawn as in one of C1L, C2L, C2sL, and C3L shown in Figure 8, and in each of these cases it contains a horizontal segment.

Case B. $\mathcal{V}_i$ is a chain. Note first that **P4**(a–b) hold for each pair of consecutive vertices u and v such that both of them are in $\mathcal{V}_i$, since all edges of $\mathcal{V}_i$ contain a horizontal segment. If $u \in \mathcal{V}_j$ and $v \in \mathcal{V}_i$ (resp., $u \in \mathcal{V}_i$ and $v \in \mathcal{V}_j$) with $j < i$, then $v = v^i_1$ (resp., $u = v^i_l$), and a similar argument as for the singletons can be applied. This ends the case analysis and completes the proof. □

Construction of Γ_K. We now show how to add $\mathcal{V}_K = \{v_n\}$ to Γ_{K-1} so as to obtain a valid drawing of G_K, and hence the desired drawing of G after replacing dummy vertices with crossing points. Recall that (v_1, v_n) is an edge of G by the definition of canonical ordering. We distinguish whether v_n is real or dummy. Note that if v_n is dummy, then its four neighbors are all real and hence their N, NW, and NE ports are free by **P6**, so we can draw v_n as depicted in Figure 12a. If v_n is real, then it has three neighbors in Γ_{K-1}. Vertex v_1 is real by construction, but the other two neighbors can be dummy. If these neighbors are real, then they are attached as depicted in Figure 12b; if one of them is dummy (or both), then we distinguish between the same cases as earlier, as shown in Figure 14. Finally, since Γ_{K-1} is attachable, we use Lemma 2 to avoid crossings and to find a suitable point to place v_n. A complete drawing is in Figure 3d.

Embeddings with no uncrossed edge. Recall that we assume that there is some edge (v_1, v_2) in the planarization of G^* such that both v_1 and v_2 are real. However, there are 1-planar drawings of 3-connected 1-planar graphs such that every edge is crossed and the planarization is also 3-connected. In this case, we do the following; see Figure 13. We choose any two crossing edges (v_1, v_b) and (v_2, v_a), remove them and add two dummy edges (v_1, v_2) and (v_a, v_b) in a planar way to obtain the graph H. We construct a valid drawing of H where we use as outer face the face containing the edge (v_1, v_2) but not (v_a, v_b). Γ_2 is drawn as depicted in Figure 13c, i.e., the edge (v_a, v_b) is a horizontal segment. After constructing a drawing of H, we replace (v_1, v_2) and (v_a, v_b) by the original edges as shown in Figure 13d. This gives a valid drawing of the original graph.

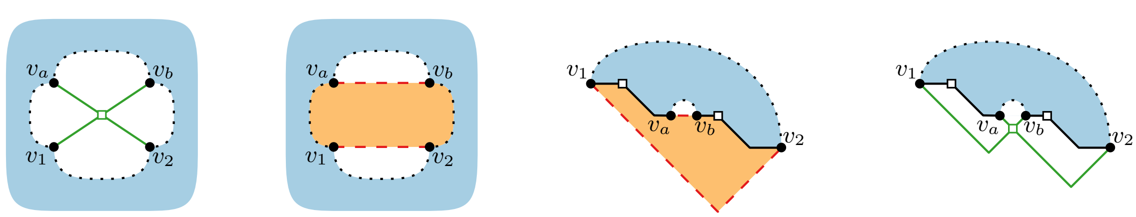
Theorem 1 follows by observing that the minimum angle between two consecutive edge segments around a crossing or around a vertex is at least $\frac{\pi}{4}$.

4 2-bend Drawings of Subcubic 1-planar Graphs

In this section, we show how to compute orthogonal drawings of all subcubic 1-planar graphs by using (at most) two bends per edge.

Theorem 2 *Every subcubic 1-plane graph admits a 2-bend 1-planar drawing using only slopes in the set* $\{0, \frac{\pi}{2}\}$ *and having both angular and crossing resolution at least* $\frac{\pi}{2}$.

Liu et al. [34] presented an algorithm to compute orthogonal drawings for planar graphs of maximum degree four with at most two bends per edge (except the octahedron, which requires three bends on one edge). We make use of their algorithm for handling 2-connected graphs. The algorithm chooses two vertices s and t and computes an st-ordering of the input graph $G = (V, E)$. Let $V = \{v_1, \ldots, v_n\}$ with $\sigma(v_i) = i$, $1 \leq i \leq n$. Liu et al. now compute an embedding of G such



(a) (b) (c) (d)

Figure 13: Illustration for the case that there is no uncrossed edge.

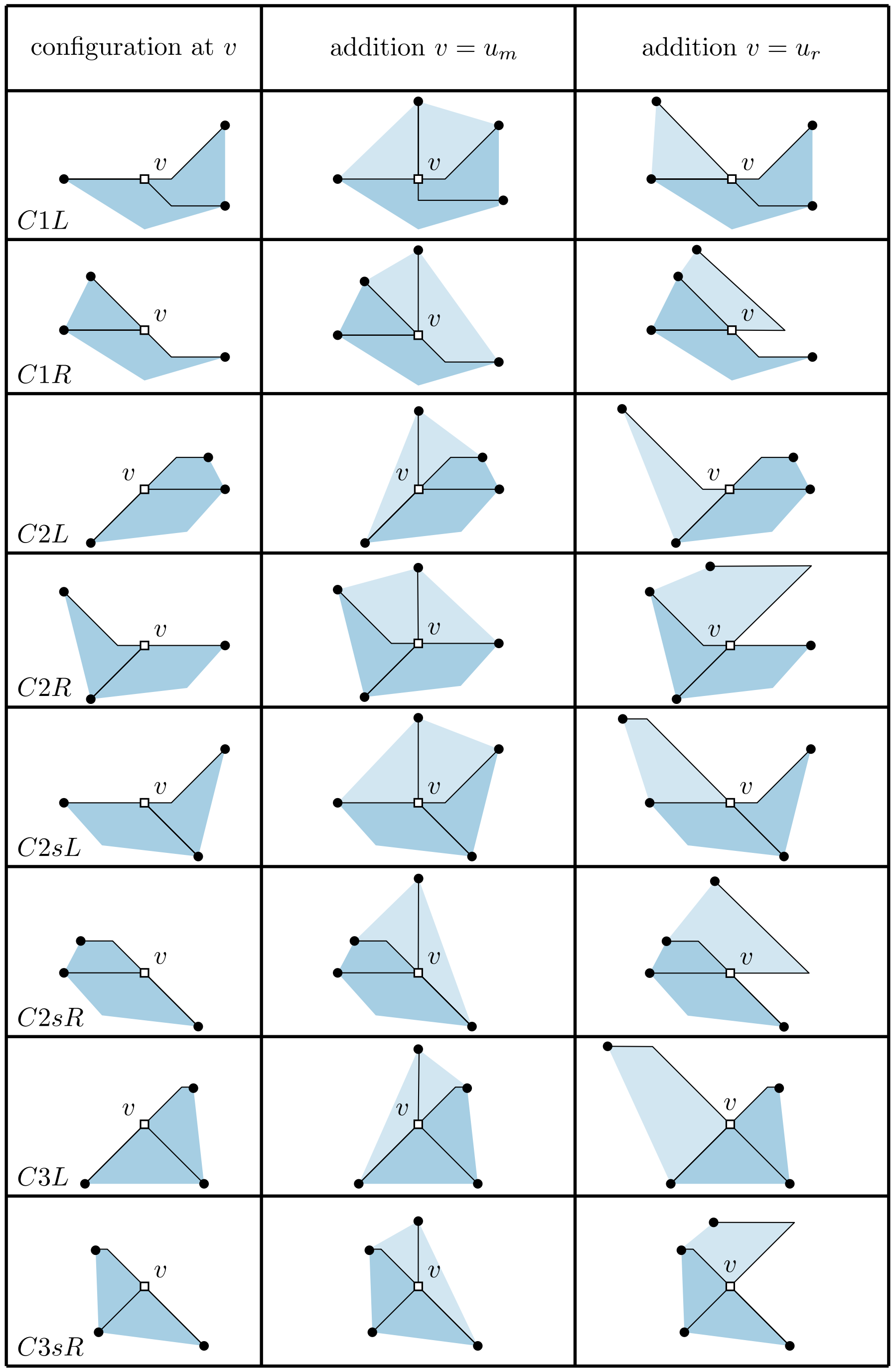


Figure 14: Cases for dummy vertices that are adjacent to v_n. The middle neighbor of v_n is u_m. The regions with a dark blue background are inner faces of the drawing before the addition of v_n; the regions with a light blue background are inner faces created by the addition of v_n.

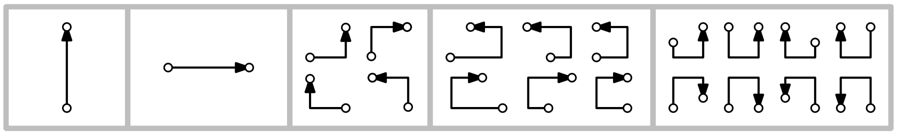


Figure 15: The shapes to draw edges.

that v_2 lies on the outer face if $\deg(s) = 4$ and v_{n-1} lies on the outer face if $\deg(t) = 4$; such an embedding exists for every graph with maximum degree four except the octahedron.

In the algorithm by Liu et al., the edges around each vertex $v_i, 1 \leq i \leq n$ are assigned to the four ports as follows. If v_i has only one outgoing edge, it uses the N port; if v_i has two outgoing edges, they use the N and E port; if v_i has three outgoing edges, they use the N, E, and W port; and if v_i has four outgoing edges, they use all four ports. Symmetrically, the incoming edges of v_i use the S, W, E, and N port, in this order. Note that every vertex except s and t has at least one incoming and one outgoing edge; hence, the given embedding of the graph provides a unique assignment of edges to ports. The edge (s, t) (if it exists) is assigned to the W port of both s and t. If $\deg(s) = 4$, the edge (s, v_2) is assigned to the S port of s (otherwise the port remains free); if $\deg(t) = 4$, the edge (t, v_{n-1}) is assigned to the N port of t (otherwise the port remains free).

Finally, they go through all vertices as prescribed by the *st*-ordering, placing them bottom-up. The way an edge is drawn is determined completely by the port assignment, as depicted in Figure 15.

Let $G = (V, E)$ be a subcubic 1-plane graph. We first re-embed G according to Lemma 1. Let G^* be the planarization of G after the re-embedding. Then, all cutvertices of G^* are real vertices, and since they have maximum degree three, there is always a bridge connecting two 2-connected components. Let $G_1, \ldots, G_k$ be the 2-connected components of G, and let G_i^* be the planarization of $G_i, 1 \leq i \leq k$. We define the *bridge decomposition tree* $\mathcal{T}$ of G as the graph having a node for each component G_i of G, and an edge (G_i, G_j), for every pair G_i, G_j connected by a bridge in G. We root $\mathcal{T}$ in G_1. For each component $G_i, 2 \leq i \leq k$, let u_i be the vertex of G_i connected to the parent of G_i in $\mathcal{T}$ by a bridge and let u_1 be an arbitrary vertex of G_1. We will create a drawing Γ_i for each component G_i with at most two slopes and two bends such that u_i lies on the outer face.

To this end, we first create a drawing Γ_i^* of G_i^* with the algorithm of Liu et al. [34] and then modify the drawing. Throughout the modifications, we will make sure that the following invariants hold for the drawing Γ_i^*.

(I1) Γ_i^* is a planar orthogonal drawing of G_i^* and its edges are drawn as in Figure 15;
(I2) u_i lies on the outer face of Γ_i^* and its N port is free;
(I3) every edge is y-monotone from its source to its target (which excludes edges drawn with U-shapes);
(I4) every edge with two bends is a C-shape, there are no edges with more bends;
(I5) if a C-shape ends in a dummy vertex, it uses only E ports; and
(I6) if a C-shape starts in a dummy vertex, it uses only W ports.

Note that the algorithm by Liu et al. has additional invariants about the exact ports used at each vertex based on the number of incoming and outgoing edges; these invariants do not hold throughout our modifications, so it can happen that, e.g., the S or the N port of a vertex is free, or an outgoing edge uses the W port instead of the E port.

Lemma 5 *Every G_i^* admits a drawing Γ_i^* that satisfies invariants (I1)–(I6).*

Proof: By construction, G_i^* is biconnected. First, observe that every face in G_i^* contains at least two real vertices since no two dummy vertices can be adjacent. Hence, there is some face that contains the real vertex $t = u_i$ and some real vertex s. We use these two vertices to compute an st-order σ and use the algorithm of Liu et al. to obtain the drawing Γ_i^* of G_i^*. We now show that this drawing satisfies all invariants. Invariants (I1) and (I2) are trivially satisfied.

Since s and t are real vertices, both have degree at most 3. All other vertices have at least one incoming and one outgoing edges. Hence, no incoming edge can use the N port and no outgoing edge can use the S port. So there are no U-shapes in the drawing; since U-shapes are the only edges that are not drawn y-monotone from its source to its target, this satisfies (I3). By construction, all edges in G_i^* are drawn with at most 2 bends; hence, invariant (I4) holds.

Consider a dummy vertex v in G_i^* with neighbors a, b, c, d in clockwise order; hence, the edges (a, c) and (b, d) cross in the given embedding of G_i. Assume w.l.o.g. that (v, a) uses the S port, (v, b) uses the W port, (v, c) uses the N port, and (v, d) uses the E port at v. Since there are no U-shapes in the drawing, both (v, a) and (v, c) have to be drawn as a vertical or an L-shape, so they both have at most 1 bend.

Consider now the edges (v, b) and (v, d). Both edges are drawn as a horizontal, an L-shape, or a C-shape. Recall that b and d are real vertices, so they have at most degree 3. If (v, b) is an outgoing edge of b, it uses the N or the E port at b, but it uses the W port at v, so it cannot be a C-shape. If (v, b) is an incoming edge of b, it uses the W or the S port at b, and it can only be a C-shape if it uses the W one. Symmetrically, if (v, d) is an outgoing edge of d, it uses the N port or the E port at d and it can only be a C-shape if it uses the E one. If (v, d) is an incoming edge of d, it uses the W or the S port at d, but it uses the E port at v, so it cannot be a C-shape. This establishes invariants (I5) and (I6) and proofs the lemma. □

We now iteratively remove the C-shapes from the drawing while maintaining the invariants. We make use of a technique similar to the stretching in Section 3. We lay an orthogonal y-monotone curve S through our drawing that intersects no vertices. Then we stretch the drawing by moving S and all features that lie right of S to the right by some distance d, and all points that lie on S are stretched to horizontal segments of length d. After this stretch, in the area between the old and the new position of S, there are no vertices, only horizontal segments of edges that are intersected by S. The same operation can be defined symmetrically for a x-monotone curve that is moved upwards.

Lemma 6 *Every G_i admits an orthogonal 2-bend drawing such that u_i lies on the outer face and its N port is free.*

Proof: We start with a drawing Γ_i^* of G_i^* that satisfies invariants (I1)–(I6), which exists by Lemma 5. By (I2), u_i lies on the outer face and its N port is free. If no dummy vertex in Γ_i^* is incident to a C-shape, by (I4) all edges incident to dummy vertices are drawn with at most 1 bend, so the resulting drawing Γ_i of G_i is an orthogonal 2-bend drawing. Otherwise, there is a C-shape between a real vertex u and a dummy vertex v. We show how to eliminate this C-shape without introducing new ones while maintaining all invariants.

We first prove the case that (u, v) is directed from u to v, so by (I5) it uses only E ports. We do a case analysis based on which ports at u are free. Note that our modifications also change port assignments at real vertices, so we cannot assume that the invariants on the used ports by Liu et al. hold. So it might happen that after some modifications the N port of a vertex is free.

In particular, we will often first move the edge that uses the N port to another port and then use Case 1 to resolve the resulting configuration. Hence, we present this case first, although it cannot happen in the very first step of the algorithm.

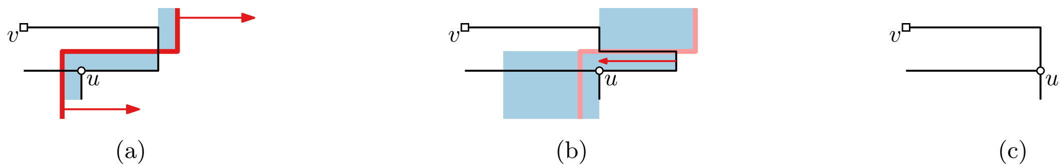


Figure 16: Proof of Lemma 6, Case 1.

Case 1. The N port at u is free; see Figure 16. Create a curve S as follows: Start at some point p slightly to the top left of u and extend it downward to infinity. Extend it from p to the right until it passes the vertical segment of (u, v) and extend it upwards to infinity. Place the curve close enough to u and (u, v) such that no vertex or bend point lies between S and the edges of u that lie right next to it. Then, stretch the drawing by moving S to the right such that u is placed below the top-right bend point of (u, v). Since S intersected a vertical segment of (u, v), this changes the edge to be drawn with four bends. However, now the rectangular region between u and the second bend point of (u, v) is empty and the N port of u is free, so we can make an L-shape out of (u, v) that uses the N port at u. This does not change the drawing style of any edge other than (u, v), so all the invariants are maintained and the number of C-shapes is reduced by one.

Case 2. The N port at u is used by an edge (u, w) and the W port is free. We distinguish three more cases based on the drawing style of (u, w).

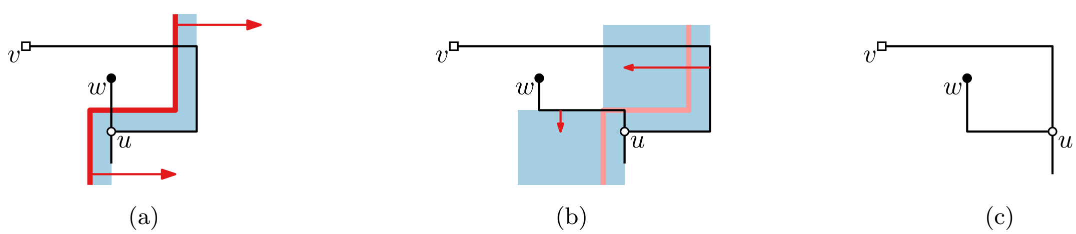


Figure 17: Proof of Lemma 6, Case 2.1.

Case 2.1. (u, w) is a vertical edge; see Figure 17. We create a curve S as in Case 1 except that we do not pass the vertical segment of (u, v) but extend it upwards to infinity before. We stretch the drawing by moving S to the right such that u is placed below the top-right bend point of (u, v). Now the edge (u, w) is drawn with 2 bends, but the area between u and the two bend points is empty and the W port of u is unused, so we can make an L-shape out of (u, w) that uses the W port at u. Furthermore, similar to Case 1, the rectangular region between u and the top-right bend point of (u, v) is free and now the N port of u is unused, so we can make an L-shape out of (u, v) that uses the N port at u.

Case 2.2. (u, w) is an L-shape and w lies to the left of u; see Figure 18. Assume first that w lies below v. We claim that there is no vertex in the rectangular region bounded by the bend point of (u, w) and the bottom bend point of (u, v). Assume to the contrary that there is some vertex in this region and let x be the bottom-most one. Since G_i is a directed s-t-graph and all edges

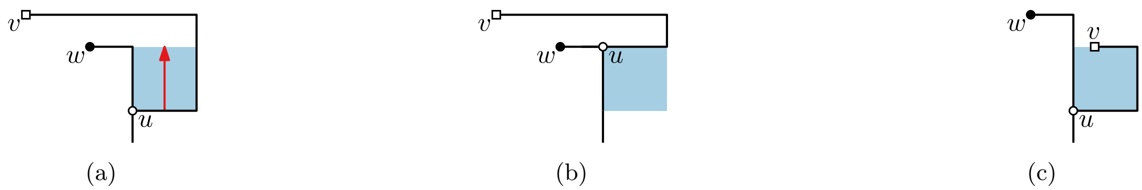


Figure 18: Proof of Lemma 6, Case 2.2.

are drawn y-monotone from their source to their target, there has to a y-monotone path from s to x. Since the E and the N port of u are already used, the only way to enter this region is by intersecting (u, w) or (u, v), which contradicts invariant (I1).

Hence, this region is empty and we can move u upwards to the same y-coordinate as w. Now (u, w) uses the W port at u and we can use Case 1 to make (u, v) an L-shape.

On the other hand, if w does not lie below v, we can use the same argument that the area described above, but bounded from the top by the y-coordinate of v, is empty. However, there has to be an edge that uses the S port of v, and it has to be y-monotone by invariant (I3), so its source has to lie below v; a contradiction.

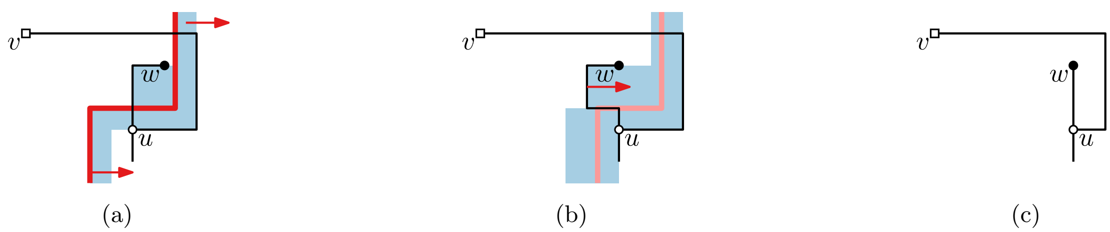


Figure 19: Proof of Lemma 6, Case 2.3.

Case 2.3. (u, w) is an L-shape and w lies to the right of u; see Figure 19. By the same argument as in Case 2.2, w has to lie below v. We can also use the exact argument to show that the rectangular region between (u, w) and (u, v) is empty.

We create a curve S as in Case 2.1. We stretch the drawing by moving S to the right such that u is placed directly below w. Because of the empty region, we can now make (u, w) vertical and then use Case 2.1 to make (u, v) an L-shape.

Case 3. The N port and the W port at u are used. Let (u, w) be the edge that uses the W port at u; since (u, v) is an outgoing edge and the edge at the N port has to be outgoing by invariant (I3), (u, w) is an incoming edge at u. By invariants (I3) and (I5), it has to be drawn as a horizontal segment or as an L-shape such that w lies below u. We distinguish two more cases based on the drawing style of (u, w).

Case 3.1. (u, w) is an L-shape and w lies below u; see Figure 20. We create a curve S as follows: We start at some point p slightly to the top left of w and extend it downward to infinity. Then we extend it from p to the right until it passes u and extend it upwards to infinity. We place the curve close enough to (u, w) such that no vertex or bend point lies between S and (u, w). Then, we stretch the drawing by moving S to the right such that w is placed below u. After this operation, the S port of u is free and there is no edge or vertex on the vertical segment between u and w, so we can make (u, w) a vertical edge and then use Case 2 to make (u, v) an L-shape.

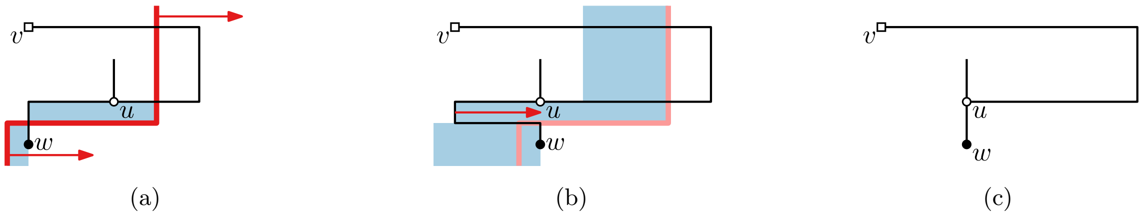


Figure 20: Proof of Lemma 6, Case 3.1.

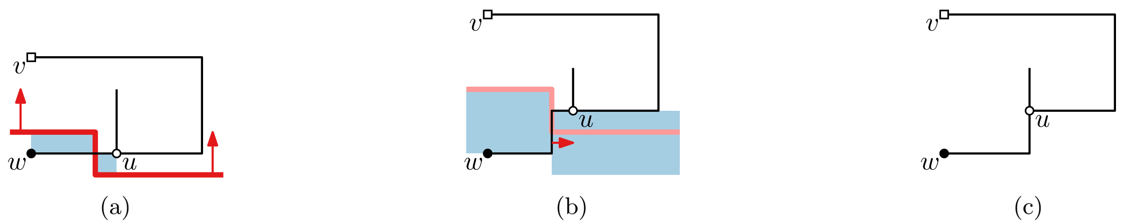


Figure 21: Proof of Lemma 6, Case 3.2.

Case 3.2. (u, w) is a horizontal edge and w is at the same y-coordinate as u; see Figure 21. We now create an x-monotone curve S as follows: We start at some point p slightly to the top left of u and extend it leftward to infinity. Then we extend it from p to the bottom until it passes (u, w) and extend it rightwards to infinity. We place the curve close enough to (u, w) such that no vertex or bend point lies between S and (u, w). Then, we stretch the drawing by moving S upwards for a short distance. After this operation, the S port of u is free and the whole rectangular region between w and u is empty, so we can make (u, w) an L-shape and then use Case 2 to make (u, v) an L-shape.

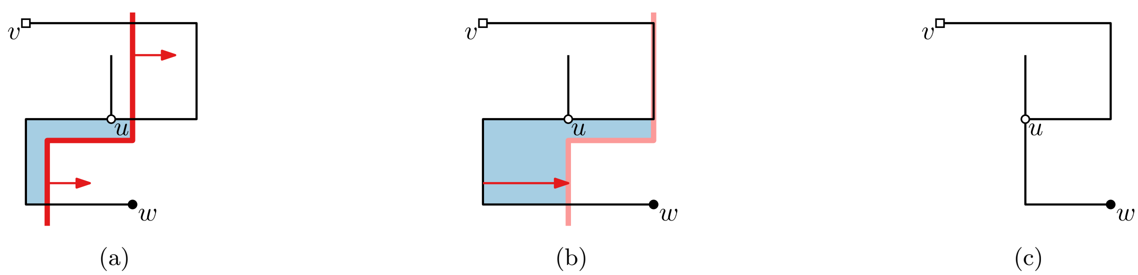


Figure 22: Proof of Lemma 6, Case 3.3.

Case 3.3. (u, w) is a C-shape; see Figure 22. We create a y-monotone curve S as follows: We start at some point p slightly to the bottom right of u and extend it upward to infinity. Then we extend it from p to the left, following the upper horizontal segment of (u, w) until it reaches the vertical segment of (u, w) and then extend it downwards to infinity. We place the curve close enough to (u, w) such that no vertex or bend point lies between S and (u, w). Then, we stretch the drawing by moving S to the right such that the whole rectangle between u and the bottom bend of (u, w) is empty. Because of the empty area, we can now make (u, w) an L-shape that uses the S port at u. Hence, the W port of u is now free and we can use Case 2 to make (u, v) an L-shape.

The case in which (u, v) is directed from v to u is completely symmetric: rotate the drawing by 180°, then (u, v) is again directed from u to v and uses only E ports. We only have to make sure that we do not use the S port of u if $u = u_i$, which would violate invariant (I2) after rotating back. This can only happen in Case 3; however, since u_i has degree 2 and the E port is used by (u_i, v), we are in Case 1 or 2, in which the S port is untouched, so invariant (I2) holds.

To conclude, we note that each of the above operations maintains all the invariants. Hence, by repeating them for every C-shape, we obtain the desired drawing of G_i. □

Finally, we combine the drawings Γ_i to a drawing Γ of G. Recall that every cutvertex is real and two biconnected components are connected by a bridge. Let G_j be a child of G_i in the bridge decomposition tree. We have drawn G_j with u_j on the outer face and a free N port. Let v_i be the neighbor of u_j in G_i. We choose one of its free ports, rotate and scale Γ_j such that it fits into the face of that port, and connect u_j and v_i with a vertical or horizontal segment. Doing this for every biconnected component gives an orthogonal 2-bend drawing of G and hence concludes the proof of Theorem 2.

5 Lower Bounds for 1-plane Graphs

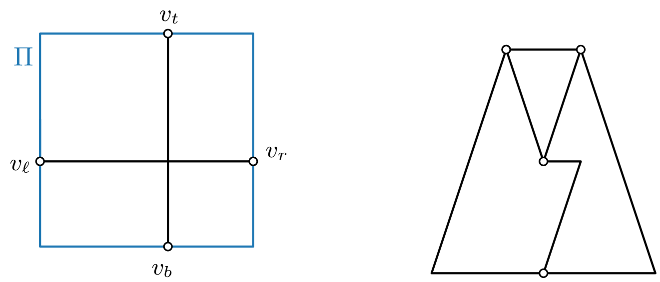
In this section, we provide lower bounds on the slope number for 1-bend and straight-line 1-planar drawings of 1-plane graphs. A 1-plane graph is specified by the combinatorial embedding of the planarization of some 1-planar drawing.

5.1 1-bend Drawings

The first lower bound we prove holds for embedding-preserving 1-bend drawings of 3-connected subcubic 1-plane graphs. The gap between this lower bound and upper bound of Theorem 1 is only one unit, however, the algorithm used to prove Theorem 1 may use a different 1-planar embedding of the input graph.

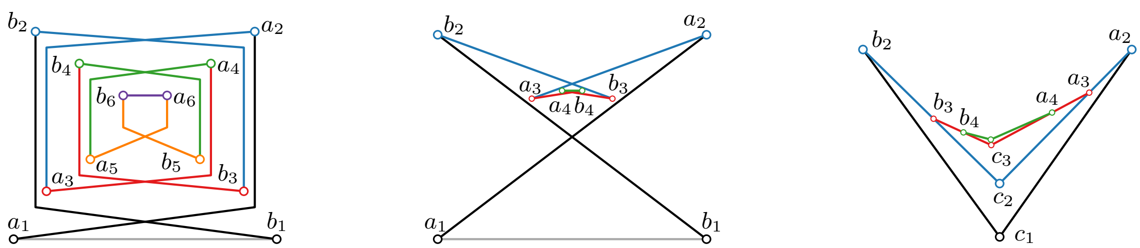
Theorem 3 *There exists a* 3*-connected subcubic* 1*-plane graph such that any embedding-preserving (fixed edge order around vertices)* 1*-bend drawing uses at least three distinct slopes.*

Proof: We show that any drawing of K_4 that uses only two slopes has a pair of crossing edges. Let G be K_4 and assume that two slopes suffice. Now consider the boundary of the drawing, which defines a polygon Π. Note that Π has at least four convex corners since otherwise we would need three slopes. Every convex corner has to use two consecutive ports, hence every convex corner is a degree-2 vertex and must be a bend bend point. Consider the set X consisting of the leftmost, rightmost, bottommost and topmost segments of Π (a segment is a consecutive set of edges with the same slope). The endpoint of any of those segments has to be a bend point (no crossing and no vertex, due to the used ports). Since on a segment no bend can be adjacent to another bend or a crossing, every segment of X contains one of the vertices of G in the interior. Let v_t be the vertex on the topmost segment and v_b be the vertex on the bottommost segment. Note that v_t and v_b have the same x-coordinate since otherwise we need two bends for their connecting edge. For the same reason the two other vertices (v_ℓ, v_r) have the same y-coordinate. As a consequence, the edges connecting v_t and v_b, and the edges connecting v_r and v_ℓ are bendless and cross each other. The remaining four edges have the bends appearing on Π and thus form the boundary. This is the only possibility for a 1-bend 1-planar drawing of K_4 with two slopes; see Fig. 23a. Combinatorially, this



(a) (b)

Figure 23: (a) The only way to draw K_4 in a 1-planar way and with at most 1 bend per edge with two slopes up to affine transformations. (b) A different embedding using three slopes.



(a) (b) (c)

Figure 24: The construction for Theorem 4: (a) The graph G_5; (b) a straight-line drawing of G_3; (c) the tower of the W_is (schematic).

is a drawing of the nonplanar embedding of K_4. Thus, for the planar embedding of G no 1-bend 1-planar drawing with 2 slopes exists. As a remark, for any embedding of K_4 three slopes suffice as indicated by Fig. 23b. □

5.2 Straight-line Drawings

We now give lower bounds for bendless drawings. We exhibit 1-plane graphs with vertex degree two such that any embedding-preserving straight-line drawing uses a number of bends that grows linearly with the size of the graph. This construction is the same as the one used by Hong et al. [24] to prove an exponential area lower bound for embedding-preserving straight-line drawings of 1-plane graphs, with two edges added to make the graph 2-connected.

Theorem 4 *There exist 2-connected 2-regular 1-plane graphs with n vertices such that any embedding-preserving straight-line drawing.*

Proof: Let G_k be the plane graph given by the planarization of the drawing Γ_k of the cycle $C_k = a_1 \dots, a_{k+1}, b_{k+1}, \dots, b_1, a_1$ as shown in Figure 24a. We denote the crossing between two edges $a_i a_{i+1}$ and $b_i b_{i+1}$ with c_i. Let W_i be the quadrilateral face of G_k with vertices c_{i-1}, a_i, c_i and b_i. Note that the edges in W_i incident to c_i and the edges in W_{i+1} incident to c_i have the

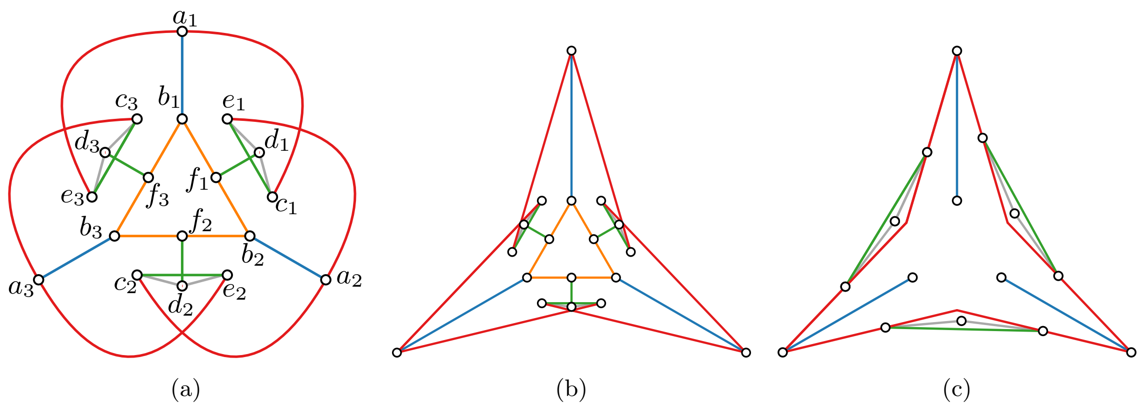


Figure 25: The construction for Lemma 7: (a) The graph G; (b) a straight-line drawing of G; (c) rearrangement of G (schematic).

same (pair of) slopes in any straight-line drawing of C_k. Hence, in any straight-line embedding-preserving redrawing of Γ_k we can rotate W_{i+1} by π around c_i and align it with W_i such that the edges meeting in c_i of both quadrilaterals overlap. The rotation will maintain the slopes. We do this for every W_i and end up with a sequence of disjoint copies of all W_i (see Figure 24c). This gives a different drawing, but it uses the same slopes. (We changed the drawing to make the analysis easier.) For all $1 < i < k$ the supporting lines of a_ic_i and $a_{i+1}c_{i+1}$ differ by rotation (angle $< \pi$) in the same direction. The total rotation of those edges cannot exceed π, since when radially sweeping from a_ic_i to $a_{i+1}c_{i+1}$ we cannot pass the slope of the (imaginary) edge b_2a_2. As a consequence, the slopes of all edges a_ic_i are different and thus also all the slopes of the edges a_ia_{i+1} have to be different. □

Note that the graphs used for Theorem 4 cannot be augmented to become 3-connected, unless either we modify their embedding or we cross an edge twice. Hence, we now study lower bounds for 3-connected 1-plane graphs. We first exhibit cubic graphs that require a relatively large number of slopes (namely, 18), and then extend this construction to general 3-connected 1-plane graphs with maximum vertex degree Δ, for which we prove a lower bound linear in Δ.

Lemma 7 *There exist 3-connected cubic 1-plane graphs such that any embedding-preserving straight-line drawing uses at least eighteen distinct slopes.*

Proof: Consider the graph G depicted in Figure 25a–b. To simplify the analysis we exploit a similar idea as in Theorem 4. Let x_i be the crossing between a_ic_i and $a_{i+1}e_i$ (indices modulo 3). Fix any straight-line drawing of G and let T_i be the triangle $e_ic_ix_i$ including the two segments e_id_i and c_id_i. For $i = 1, 2, 3$ we cut T_i, rotate it by π around x_i and put it back to the drawing. This constructs a new drawing with the same set of slopes. The modified drawing contains a pseudo-triangle (polygon with exactly three convex corners), whose chains ($a_ic_id_ie_ia_{i+1}$) have four edges (see Figure 25c). Further, for every chain there is an edge (e_ic_i) between the second and fourth vertex cutting off d_i. The edges of a pseudo-triangle have different slopes. Thus, we have 12 different slopes here. Moreover, if you traverse the edges of a pseudo-triangle in cyclic order they will be ordered by slope. Since a_ib_i is in between a_ic_i and a_ie_{i+2} when doing a radial sweep we have three more distinct slopes. Finally, we note that replacing the edges e_id_i and c_id_i with

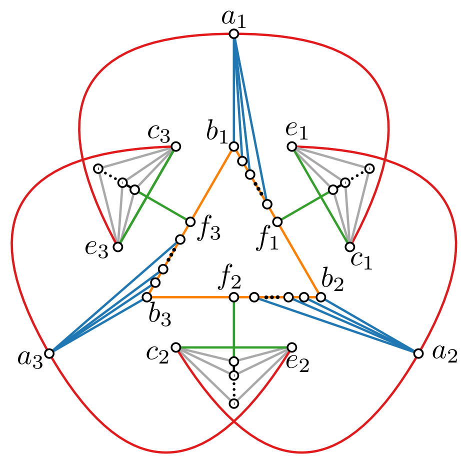


Figure 26: The construction for Theorem 5.

e_ic_i gives another pseudo triangle that avoids the slopes of e_id_i and c_id_i. As a consequence, the edges e_ic_i will give us three new slopes and we end up with eighteen different slopes.

To obtain an infinite family of graphs, observe that we do not use the edges between b- and f-vertices in our analysis. Hence, we can subdivide the edges (b_1, f_3) and (b_2, f_1) several times and connect pairs of subdivision vertices. □

Theorem 5 *There exist* 3*-connected* 1*-plane graphs with maximum vertex degree* Δ *such that any embedding-preserving straight-line drawing uses at least* $9(\Delta - 1)$ *distinct slopes.*

Proof: Consider the graph in Figure 26. The degree of a_i, c_i and e_i is Δ.

We can repeat the argument of the proof of Lemma 7. There are only two differences: (i) instead of a single edge a_ib_i there is a bundle of edges incident to a_i. However the whole bundle lies in between a_ic_i and a_ie_{i+2} and therefore the slopes of these edges are distinct. (ii) Instead of the pseudo-triangle with chains $a_ic_id_ie_ia_{i+1}$ we have now a sequence of nested chains given by the edges incident to e_i and c_i. All these "subchains" are contained in the triangle $e_ic_ix_i$, where x_i is the crossing between a_ic_i and $a_{i+1}e_i$. Their slopes lie between the slopes of a_ic_i and e_ia_{i+1}. This means that the three edge-bundles of subchains are separated by slopes. Clearly the slopes within each bundle have to be different.

Counting the slopes, we have the 18 slopes of the subgraph shown in Figure 25a and then there are 9 vertices, each incident to $\Delta - 3$ new edges, that will need a new slope. In total we need $18 + 9(\Delta - 3) = 9(\Delta - 1)$ distinct slopes.

To obtain an infinite family of graphs, we can again subdivide (b_1, f_3) and (b_2, f_1) several times and connect pairs of the subdivision vertices. □

6 Open problems

The research in this paper gives rise to interesting questions, in what follows we mention some of them.

Concerning 1-bend drawings of 1-planar graphs, a natural question is whether Theorem 1 can be extended to all subcubic 1-planar graphs. Observe that it may not be possible to augment a 2-connected 1-planar graph in order to make it 3-connected without loosing 1-planarity [9].

Our lower bounds for straight-line drawings hold only for embedding-preserving drawings. It would be very interesting to devise lower bounds that hold with the only restriction that the drawing be 1-planar. A challenge to face in this direction is that 1-planar graphs may have exponentially many embeddings even when 4-connected [8].

Finally, a more general question is whether the 1-planar slope number of 1-planar graphs is bounded by a function of the maximum vertex degree, which would extend the result by Keszegh et al. [28] for planar graphs.

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