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Efficient Generation of Different Topological Representations of Graphs Beyond-Planarity

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Abstract

Beyond-planarity focuses on combinatorial properties of classes of non-planar graphs that allow for representations satisfying certain local geometric or topological constraints on their edge crossings. Beside the study of a specific graph class for its maximum edge density, another parameter that is often considered in the literature is the size of the largest complete or complete bipartite graph belonging to it.

Overcoming the limitations of standard combinatorial arguments, we present a technique to systematically generate all non-isomorphic topological representations of complete and complete bipartite graphs, taking into account the constraints of the specific class. As a proof of concept, we apply our technique to various beyond-planarity classes and achieve new tight bounds for the aforementioned parameter.

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1 Introduction

Beyond-planarity is an active research area concerned with combinatorial properties of non-planar graphs that somehow lie in the "neighborhood" of planar graphs. More concretely, these graphs allow for non-planar drawings in which certain geometric or topological crossing configurations are forbidden. The most studied beyond-planarity graph classes, with early results dating back to 60's [13, 53], are the *k-planar* graphs [50], which forbid an edge to be crossed more than k times, and the *k-quasiplanar* graphs [5], which forbid k mutually crossing edges; for an illustration refer to Figs. 1a-1b.

More recently, several other graph classes have been suggested in the literature (see, e.g., [3, 9, 15, 24]), also motivated by cognitive experiments [42, 48] indicating that the absence of certain types of crossings helps in improving the readability of a drawing of a graph; for a survey, we point the reader to [32]. Some of the most studied such graph classes are:

- *fan-planar* graphs, in which no edge can be crossed by two independent edges or by two adjacent edges from different directions [16, 17, 18, 43]; e.g., in the left part of Fig. 1c the vertically drawn edge is crossed by two independent edges, which is forbidden by fan-planarity, while in its right part the vertically drawn edge is crossed by two edges from different directions, which is also not allowed by fan-planarity,

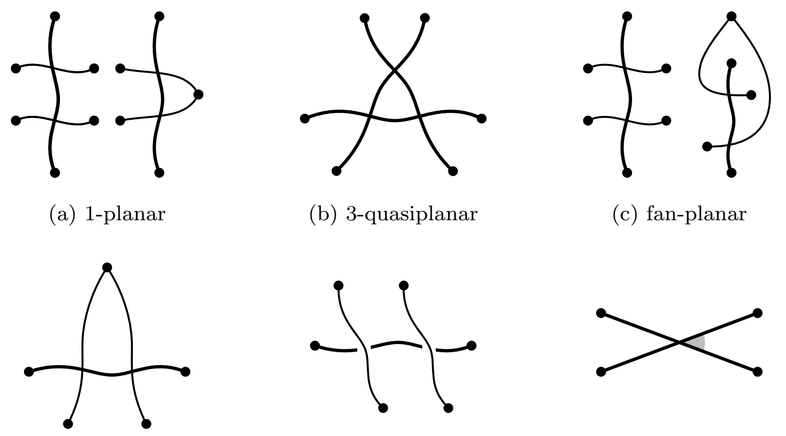
- *fan-crossing free* graphs, in which no edge can be crossed by two adjacent edges [23, 27]; e.g., in Fig. 1d the horizontally drawn edge is crossed by edges incident to a common vertex (i.e., forming a fan), which is forbidden by fan-crossing freeness,

- *gap-planar* graphs, in which each crossing is assigned to one of its two involved edges, such that each edge can be assigned at most one crossing [15]; e.g., in Fig. 1e the horizontally drawn edge has been assigned two crossings (represented as gaps), which is forbidden by gap-planarity, and

- *RAC graphs*, in which edge crossings occur only at right angles [30, 31, 33]; e.g., Fig. 1f illustrates a crossing between two edges that is not allowed since the formed angle is clearly less than 90°.

Two notable subclasses of 1-planar graphs are the *IC-planar* [8, 57] and *NIC-planar* [56] graphs, in which the crossings are independent (i. e., no two pairs of crossing edges share a vertex) and nearly independent (i. e., any two pairs of crossing edges share at most one vertex), respectively. We also remark that if one relaxes the second restriction in the definition of fan-planar graphs (i.e., the one on the right part of Fig. 1c, which concerns the direction of the crossings), then the resulting graph class is a proper super-class of the fan-planar graphs, whose members are referred to as *fan-crossing* graphs [21, 22].

Furthermore, it is worth mentioning that all the aforementioned graph classes are *topological*, i.e., each edge is represented as a simple curve, with the only exception of the class of RAC graphs, which is a purely *geometric* graph class,



(a) 1-planar (b) 3-quasiplanar (c) fan-planar

(d) fan-crossing free (e) gap-planar (f) RAC

Figure 1: Different forbidden crossing configurations.

i.e., each edge must be represented as a straight-line segment. In this work, we refer to the aforementioned topological graph classes as *beyond-planarity classes of topological graphs*.

A common characteristic of these graph classes is that their edge density is at most linear in the number of vertices, e.g., 1-planar graphs with n vertices have at most $4n - 8$ edges [50]; the known density bounds for several graph classes are provided in Table 1. Another common measure to determine the extent of a specific class is the size of the largest complete or complete bipartite graph belonging to it [15, 20, 28, 29], which also provides a lower bound on their chromatic number [40] and has been studied in related fields (refer, e.g., to [12, 25, 19, 34, 35, 41, 54]).

For 1-planar graphs, Czap and Hudák [28] proved that the complete graph K_n is 1-planar if and only if $n \leq 6$, and that the complete bipartite graph $K_{a,b}$, with $a \leq b$, is 1-planar if and only if $a \leq 2$, or $a = 3$ and $b \leq 6$, or $a = b = 4$. Analogous characterizations are known for the classes of IC-planar, NIC-planar and RAC graphs. In fact, the complete graph K_n belongs to any of these classes of graphs if and only if $n \leq 5$ [30, 56, 57]. On the other hand, the complete bipartite graph $K_{a,b}$, with $a \leq b$, is IC-planar if and only if $b \leq 3$ [56], and NIC-planar or RAC if and only if $a \leq 2$, or $a = 3$ and $b \leq 4$ [29, 56]. For the classes of 3-quasiplanar (also known as *quasiplanar*), gap-planar, and fan-crossing free graphs, characterizations exist only for complete graphs, i.e., K_n is quasiplanar if and only if $n \leq 10$ [4, 20], gap-planar if and only if $n \leq 8$ [15], and fan-crossing free if and only if $n \leq 6$ [27, 28]. We provide more details in Table 1.

To prove the "if part" of these characterizations, one has to provide a certificate drawing of the respective graph that respects the constraints of the specific graph class. The proof for the "only if part" is generally more complex, as it requires arguments to show that no such drawing exists.

One of the main techniques is provided by the linear edge density of the graph classes; e.g., K_7 is neither 1-planar nor fan-crossing free, as it has more than $4n-8$ edges [27, 50]. However, this technique has a limited applicability; e.g., for 2-planar and fan-planar graphs, which have at most $5n-10$ edges, it only ensures that K_9 is not a member of these classes. Proving that K_8 is also not a member requires a different approach. The limitations are even more evident for complete bipartite graphs, as they are sparser than the complete ones.

Another technique consists of showing that the minimum number of crossings required by *any* drawing of a certain graph (as derived by, e.g., the Crossing Lemma [2, 6, 7, 47, 49] or closed formulas [39, 55]) exceeds the maximum number of crossings allowed in the considered graph class. However, this technique only applies to graph classes that impose such restrictions, such as the classes of gap-planar and 1-planar graphs [14, 28].

Motivation. The difficulty in finding combinatorial arguments to prove that certain complete or complete bipartite graphs do not belong to specific graph classes often results in the need of a large case analysis on the different topological representations of the graph. Beside the proofs in [29, 44], we give in the arXiv version [11] of this paper another example of a combinatorial proof that, based on a tedious case analysis, yields a characterization of the complete bipartite fan-crossing free graphs. The range of the cases in these proofs justifies the need of a tailored approach to systematically explore them.

Our contribution. We suggest a technique to engineer the analysis of all topological representations of a graph that satisfy certain beyond-planarity constraints. Our technique is tailored for complete and complete bipartite graphs, as it exploits their symmetry to reduce the search space, by discarding equivalent topological representations. However, it does not extend to classes of geometric graphs (such as the RAC graphs), as it is strongly based on tools that build upon the topology of the graph and not the actual geometry.

In Section 2 we introduce some preliminary definitions. In Section 3, we present an algorithm to generate all possible representations of such graphs under different topological constraints on the crossing configurations. Our algorithm builds upon two key ingredients, which allow to drastically reduce the search space. First, the representations are constructed by adding a vertex at a time, directly taking into account the topological constraints, thus avoiding constructing unnecessary representations. Second, at each intermediate step, the produced drawings are efficiently tested for equivalence (up to a relabeling of the vertices), which usually allows to discard a large number of them. Using this algorithm, we derived characterizations for several classes of topological graphs beyond planarity, as described in Section 4; Table 1 positions our results with respect to the state of the art. In Section 5 we provide some statistics about the computations we performed to obtain our results. Finally, we discuss future directions in Section 6.

Table 1: Overview of the known results, combining the previous literature with our findings. For each class, we present the largest complete and complete bipartite graphs that belong to this class (col. "$\in$"), and the smallest ones that do not (col. "$\notin$"), and we indicate whether this follows from the literature (references) or from one of our results (Characterization or Observation). Color gray indicates weaker results that follow from other entries. For example, the fact that $K_{3,19}$ is not 4-planar implies that $K_{4,19}$, $K_{5,19}$, and $K_{6,19}$ are not 4-planar, either.

		complete				complete bipartite			
Class	Density	$\in$	Ref.	$\notin$	Ref.	$\in$	Ref.	$\notin$	Ref.
IC-planar	$\frac{13}{4}n-6$	K_5	[33, Fig.5]	K_6	[57, Prp.2.1]	$K_{3,3}$	[56, Cor.19]	$K_{3,4}$	[56, Cor.19]
NIC-planar	$\frac{18}{5}n-\frac{36}{5}$	K_5	[56, Thm.7]	K_6	[56, Thm.7]	$K_{3,4}$	[56, Thm.9]	$K_{3,5}$	[56, Thm.9]
						$K_{3,4}$		$K_{4,4}$	[56, Thm.9]
1-planar	$4n-8$	K_6	[28, Fig.1]	K_7	[50, Thm.1]	$K_{3,6}$	[28, Fig.2]	$K_{3,7}$	[28, Lem.4.2]
						$K_{4,4}$	[28, Fig.3]	$K_{4,5}$	[28, Lem.4.3]
2-planar	$5n-10$	K_7	[18, Fig.7]	K_8	Char.2	$K_{3,10}$	[9, Lem.1]	$K_{3,11}$	[9, Lem.1]
						$K_{4,6}$	Char.3	$K_{4,7}$	Char.3
						$K_{4,5}$		$K_{5,5}$	Char.3 [44]
3-planar	$\frac{11}{2}n-11$	K_8	Char.2	K_9	Char.2	$K_{3,14}$	[9, Lem.1]	$K_{3,15}$	[9, Lem.1]
						$K_{4,9}$	Char.4	$K_{4,10}$	Char.4
						$K_{5,6}$	Char.4	$K_{5,7}$	Char.4
						$K_{5,6}$		$K_{6,6}$	Char.4
4-planar	$6n-12$	K_9	Char.2	K_{10}	Char.2	$K_{3,18}$	[9, Lem.1]	$K_{3,19}$	[9, Lem.1]
						$K_{4,11}$	Obs.5	$K_{4,19}$	
						$K_{5,8}$	Obs.5	$K_{5,19}$	
						$K_{6,6}$	Obs.5	$K_{6,19}$	
5-planar	$< 8.52n$	K_9	Char.2	K_{10}	Char.2	$K_{3,22}$	[9, Lem.1]	$K_{3,23}$	[9, Lem.1]
fan-planar fan-crossing	$5n-10$	K_7	[18, Fig.7]	K_8	Char.6	$K_{4,n}$	[43, Fig.3]	$K_{5,5}$	Char.7
fan-crossing free	$4n-8$	K_6	[28, Fig.1]	K_7	[27, Thm.1]	$K_{3,6}$		$K_{3,7}$	Char.9
						$K_{4,6}$	Char.9	$K_{4,7}$	
						$K_{4,5}$		$K_{5,5}$	Char.9
gap-planar	$5n-10$	K_8	[15, Fig.7]	K_9	[15, Thm.23]	$K_{3,12}$	[15, Fig.7]	$K_{3,14}$	[14, Thm.1]
						$K_{4,8}$	[15, Fig.9]	$K_{4,9}$	Obs.11
						$K_{5,6}$	[15, Fig.9]	$K_{5,7}$	[15]
						$K_{5,6}$		$K_{6,6}$	[14, Thm.1]
RAC	$4n-10$	K_5	[33, Fig.5]	K_6	[30, Thm.1]	$K_{3,4}$	[29, Fig.4]	$K_{3,5}$	[29, Thm.2]
						$K_{3,4}$		$K_{4,4}$	[29, Thm.2]
quasiplanar	$\frac{13}{2}n-20$	K_{10}	[20, Fig.1]	K_{11}	[4, Thm.5]	$K_{4,n}$	[43, Fig.3]	–	
						$K_{5,18}$	Obs.13	?	
						$K_{6,10}$	Obs.13	?	
						$K_{7,7}$	Obs.13	$K_{7,52}$	[4, Thm.5]

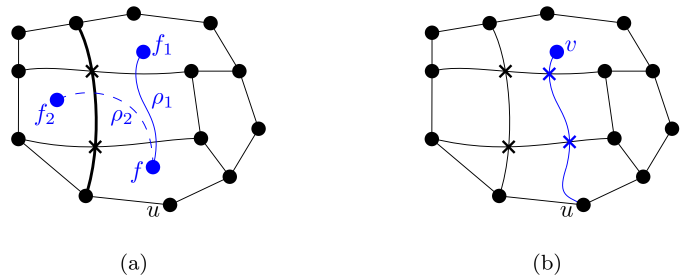


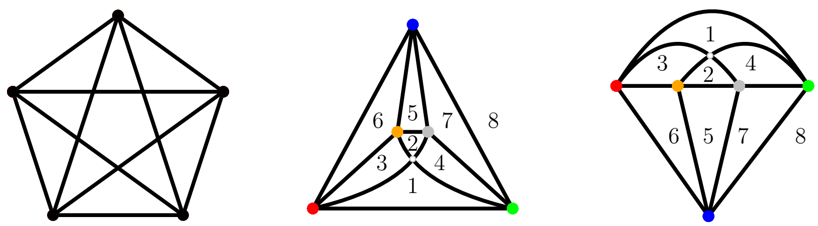
Figure 2: Illustration of (a) two pathways ρ_1 (solid blue) and ρ_2 (dashed blue) for u of length 2, with destinations f_1 and f_2 (the crosses indicate dummy vertices of Γ). For the class of 2-planar graphs, ρ_1 is valid, while ρ_2 is not valid, since in its presence the bold drawn edge has three crossings; (b) an augmentation of Γ by edge (u, v), using the valid pathway ρ_1.

2 Preliminaries

We assume familiarity with standard definitions on planar graphs and drawings. In this paper, we consider graphs containing neither multi-edges nor self-loops. Let $G = (V, E)$ be a graph. A *drawing* of G is a topological representation of G in the plane $\mathbb{R}^2$ such that each vertex $v \in V$ is mapped to a distinct point p_v of the plane, and each edge $(u, v) \in E$ is drawn as a simple Jordan curve connecting its endpoints p_u and p_v without passing through any other vertex. Unless otherwise specified, we consider *simple* drawings, in which any two edges intersect in at most one point, which is either a common endpoint or a proper crossing. Hence, no two edges are allowed to cross twice (or more times), and no two edges incident to the same vertex are allowed to cross. We note, however, that the simplicity assumption may be not without loss of generality for some of the graph classes; e.g., in the case of quasiplanar graphs [4].

A drawing without edge crossings is called *planar*. Accordingly, a graph that admits a planar drawing is called *planar*. The *planarization* of a (non-planar) drawing is the planar drawing obtained by replacing each of its crossings with a dummy vertex. The dummy vertices are referred to as *crossing vertices*, while the remaining ones (that is, the ones of the original drawing) as *real vertices*. A planar drawing partitions the plane into connected regions, called *faces*; the unbounded one is called *outer face*. The *degree* of a face is defined as the number of edges on its boundary, counted with multiplicity. The *dual* of a planar drawing Γ has a node for each face of Γ and an arc between two nodes if the corresponding faces of Γ share an edge.

Let D be a drawing of a graph G and let Γ be its planarization. A *half-pathway for a vertex u* in Γ is a path in the dual of Γ from a face incident to u to some face in Γ, called its *destination*; see Fig. 2. The *length* of a half-pathway is the number of edges in this path. A half-pathway ρ for u is *valid* with respect to a beyond-planarity class $\mathcal{C}$ of topological graphs, if Γ can be augmented in



(a) (b) (c)

Figure 3: Different drawings of K_5: The drawing of (a) is isomorphic neither to the one of (b) nor to the one of (c), while the drawings of (b) and (c) are in fact isomorphic; the colors of the vertices and the labels show the vertex and facial correspondences.

such a way that:

(i) a vertex v is placed in the interior of the destination of ρ,

(ii) edge (u, v) is drawn as a curve from u to v that crosses only the edges that are dual to the edges in ρ, in the same order, and

(iii) drawing edge (u, v) in D with the same curve as in Γ, results in a simple drawing that satisfies the restrictions of class $\mathcal{C}$.

Accordingly, a *pathway for an edge* (u, v) is a half-pathway for vertex u in Γ, whose destination is a face incident to vertex v. A *valid pathway* is defined analogously, with the only difference that v is already part of Γ.

Another ingredient of our algorithm is an equivalence-relationship between different drawings of a graph G. We say that two drawings D_1 and D_2 of G are *isomorphic* [45] if there exists a homeomorphism of the sphere transforming D_1 into D_2; see Fig. 3 for an illustration. In other words, D_1 and D_2 are isomorphic if D_1 can be transformed into D_2 by relabeling vertices, edges, and faces of D_1, and by moving vertices and edges of D_1, so that at no time of this process new crossings are introduced, existing crossings are eliminated, or the order of the crossings along an edge is modified. To determine whether two drawings are isomorphic, we make use of the following definition. A bijective mapping between vertices, crossings, edges, and faces of the planarizations Γ_1 and Γ_2 of D_1 and D_2 is *valid* if and only if the following two properties hold.

P.1 if an edge (v_1, w_1) is mapped to an edge (v_2, w_2) in Γ_1 and Γ_2, respectively, and v_1 is mapped to v_2, then w_1 is mapped to w_2;

P.2 if a face f_1 is mapped to a face f_2 in Γ_1 and Γ_2, respectively, and an edge e_1 incident to f_1 is mapped to an edge e_2 incident to f_2, then the predecessor (successor) of e_1 is mapped to the predecessor (successor) of e_2 when walking along the boundaries of f_1 and f_2 in clockwise direction. Also, the face incident to the other side of e_1 is mapped to the face incident to the other side of e_2.

To see that Properties P.1 and P.2 are necessary and sufficient for D_1 and D_2 to be isomorphic, observe that Property P.1 describes the relabeling of the vertices and edges in the definition of isomorphism, Property P.2 describes the corresponding relabeling of the faces, while the fact that the crossing configuration is preserved during the transformation is guaranteed by the fact that Properties P.1 and P.2 hold on the planarizations of the original drawings. Note that Property P.2 guarantees that two vertices are mapped to each other only if they have the same degree.

We conclude this section by mentioning that several works (see, e.g., [1, 38, 52]) that generate simple drawings of complete graphs adopt a weaker definition of isomorphism. Namely, two drawings D_1 and D_2 are *weakly isomorphic* [45], if there exists an incidence preserving bijection between their vertices and edges, such that two edges cross in D_1 if and only if they do in D_2. Weakly isomorphic drawings that are non-isomorphic differ in the order in which their edges cross [37]. Two simple drawings of a complete graph with the same cyclic order of the edges around each vertex (called *rotation system*) are weakly isomorphic, and vice versa [37, 51]; hence, generating all simple drawings of a complete graph reduces to finding all rotation systems that determine simple drawings [46]. However, this property holds only for complete graphs [1], while for the complete bipartite graphs, which are more difficult to handle, only partial results exist in the literature [26]. Thus, we decided not to follow this approach.

3 Generation Procedure

Let $\mathcal{C}$ be a beyond-planarity class of topological graphs and let G be a graph with $n \geq 3$ vertices. Assuming that G is either complete or complete bipartite[1], we describe in this section a recursive algorithm to compute a set $\mathcal{S}$ containing all non-isomorphic simple drawings of G that are certificates that G belongs to $\mathcal{C}$ (if any); refer to Algorithm 1 for an outline of the main steps of our technique. With slight abuse of terminology, in the following we will (sometimes implicitly) assume that $\mathcal{S}$ contains the planarizations of the drawings of G, since the (valid) pathways and the isomorphism between drawings are defined on the planarizations.

In the base of the recursion (see Line 10 of Algorithm 1), graph G is a cycle of length 3 or 4, depending on whether G is the complete graph K_3 or the complete bipartite graph $K_{2,2}$. In the former case, set $\mathcal{S}$ only contains a planar drawing of K_3, while in the latter case, set $\mathcal{S}$ contains a planar drawing and one with a crossing between two non-adjacent edges. This is because, in both cases, any other drawing is either isomorphic to one of these, or non-simple.

In the recursive step, we consider a vertex v of G (see Line 2 of Algorithm 1) and assume that we have recursively computed a set $\mathcal{S}'$ containing all non-isomorphic simple drawings of $G \setminus \{v\}$ (see Line 3 of Algorithm 1) that belong

[1] We stress that, if G is neither complete nor complete bipartite, then it is a more involved task to recognize isomorphic drawings [36], and thus to eliminate them, which is a key point in the efficiency of our approach (we provide more details in Section 4).

Algorithm 1: Enumeration Algorithm

Input: A complete (bipartite) graph G and a class $\mathcal{C}$ beyond planarity.
Output: All non-isomorphic drawings of G that are certificates that G belongs to $\mathcal{C}$.

ENUMERATE(*Graph:* G)
1 **if** $G \notin \{K_3, K_{2,2}\}$ **then**
2 $\quad v \leftarrow$ a vertex of G;
3 $\quad \mathcal{S}' \leftarrow$ ENUMERATE$(G \setminus \{v\})$;
4 $\quad \mathcal{S} \leftarrow \emptyset$;
5 $\quad$ **foreach** *drawing* D *in* $\mathcal{S}'$ **do**
$\quad\quad$ /* Add v and its edges to D in all possible ways respecting $\mathcal{C}$ */
6 $\quad\quad \mathcal{S} \leftarrow \mathcal{S} \cup$ INSERT$(v, \Gamma, \mathcal{C})$;
7 $\quad$ **end**
8 $\quad$ Remove drawings from $\mathcal{S}$ that are isomorphic to other ones in $\mathcal{S}$;
9 **else**
$\quad$ /* G is the complete graph K_3 or the complete bipartite graph $K_{2,2}$ */
10 $\quad \mathcal{S} \leftarrow$ all non isomorphic drawings of G;
11 **end**
12 **return** $\mathcal{S}$;

to $\mathcal{C}$. We may assume w.l.o.g. that $\mathcal{S}' \neq \emptyset$, as otherwise we can conclude that G does not belong to $\mathcal{C}$. Then, we consider each drawing of $\mathcal{S}'$ and our goal is to report all non-isomorphic simple drawings of G that have it as a subdrawing, and add them to set $\mathcal{S}$, where $\mathcal{S}$ is initially empty. In other words, we aim at reporting all non-isomorphic simple drawings that can be derived by all different placements of vertex v and the routing of its incident edges in the drawings of $\mathcal{S}'$ (see Line 6 of Algorithm 1, and also Algorithm 2, which outlines the main steps of the procedure to insert vertex v into the current drawing). To this end, let D be a drawing in $\mathcal{S}'$, let Γ be its planarization, and let $u_1, \ldots, u_k$ be the neighbors of v in G, where $k = \deg(v)$ (see Line 1 of Algorithm 2). If G is a complete graph, then $k = n-1$; otherwise, G is a complete bipartite graph $K_{a,b}$ with $a + b = n$, and $k = a$ or $k = b$ holds.

Insertion procedure. We start by computing all possible valid half-pathways for u_1 in Γ with respect to $\mathcal{C}$, which corresponds to constructing all possible drawings of edge (v, u_1) that respect simplicity and the restrictions of class $\mathcal{C}$ (see Line 3 of Algorithm 2). To compute these half-pathways, we again use recursion. For each half-pathway, we maintain a list of so-called *prohibited* edges, which are not allowed to be crossed when inserting edge (u_1, v), as otherwise either the simplicity or the crossing restrictions of class $\mathcal{C}$ would be violated, making the half-pathway not valid; see Fig. 4 and Fig. 5. This list is initialized with all edges of Γ corresponding to edges of D that are incident to u_1, and is updated at every recursive step.

In the base of this inner recursion, we determine all valid half-pathways for

Algorithm 2: Insertion Algorithm

Input: A vertex v, a drawing D, and a class $\mathcal{C}$

Output: All non-isomorphic drawings that contain v, belong to $\mathcal{C}$ and have D as a subdrawing.

INSERT(*Vertex:* v*, Drawing:* D*, Class:* $\mathcal{C}$)

1 $u_1, \ldots, u_k \leftarrow$ the neighbors of v in G;
2 $\mathcal{S}_1, \mathcal{S}_2 \leftarrow \emptyset$;
3 **foreach** *valid half-pathway* p *for* u_1 *in* D **do**
    /* choose a face for v and connect it to u_1 */
4   Insert into $\mathcal{S}_1$ the drawing obtained by inserting an edge (following p) and a new vertex v (in the destination of p) into D;
5 **end**
6 **for** $i = 2, \ldots, k$ **do**
    /* connect v to all its other neighbors */
7   **foreach** *drawing* D' *in* $\mathcal{S}_1$ **do**
8   **foreach** *valid pathway* p *for* (v, u_i) *in* D' **do**
9   Insert into $\mathcal{S}_2$ the drawing obtained by inserting an edge (following p) into D';
10   **end**
11   **end**
12     $\mathcal{S}_1 \leftarrow \mathcal{S}_2$;
13     $\mathcal{S}_2 \leftarrow \emptyset$;
14 **end**
15 **return** $\mathcal{S}_1$

u_1 of length zero; this means that, for each face f incident to u_1, we create a half-pathway that starts at f and has its destination also at f, which corresponds to placing v in f and drawing edge (v, u_1) crossing-free. Assume now that we have computed all valid half-pathways of some length $i \geq 0$ in Γ. We show how to compute all valid half-pathways for u_1 of length $i + 1$ (if any). Consider a half-pathway p of length i. Let f_p be its destination. Every non-prohibited edge e of f_p implies a new half-pathway of length $i + 1$, composed of p followed by the edge that is dual to e in Γ. After this step, we add to the set of prohibited edges all the edges of Γ that correspond to the same edge of G as e to guarantee simplicity. We also add to this set all the edges of Γ that cannot be further crossed due to the restrictions of class $\mathcal{C}$. We note at this point that this process will eventually terminate, since the length of a half-pathway is bounded by the number of edges of Γ.

For each valid half-pathway p computed by the procedure above, we obtain a new drawing by inserting (u_1, v) into Γ following p and by inserting v into the destination of p (see Line 4 of Algorithm 2). It remains to insert the remaining edges incident to v, i.e., $(v, u_2), \ldots, (v, u_k)$, into each of these drawings – again

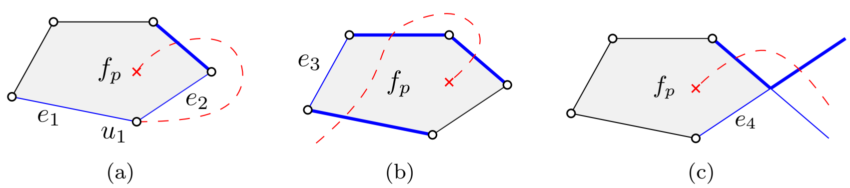


Figure 4: The prohibited edges (blue solid) for a half-pathway (red dashed) that ends in a face f_p. The thick blue edges are prohibited, because they are crossed by the half-pathway. In (a) edges e_1 and e_2 are prohibited, since they are incident to u_1. In (b) edge e_3 is prohibited, since, in order to cross this edge, the half-pathway would make a self-crossing. In (c) edge e_4 is prohibited since it is part of a crossed edge.

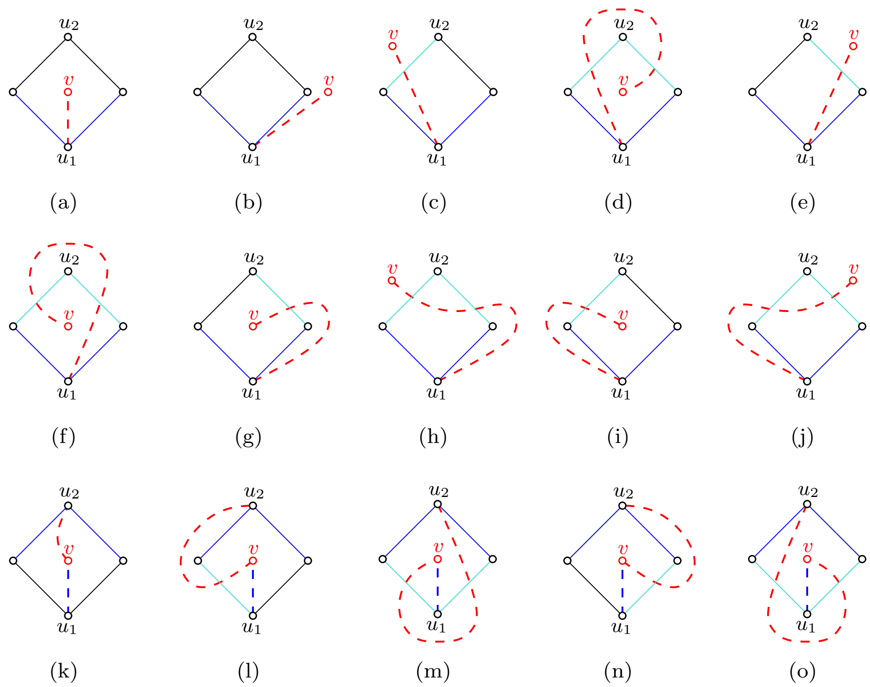


Figure 5: Illustration of an example for the insertion of a node v into a crossing-free 4-cycle, such that v is connected to two vertices u_1 and u_2. The dashed red edge is the newly inserted edge; the blue edges are prohibited; the turquoise edges are the edges that are marked as prohibited while computing the half-pathway of the red edge. Figs 5a–5j illustrate all possible ways for drawing edge (v, u_1). Figs 5k–5o illustrate all possible ways for inserting edge (v, u_2) into the drawing of Fig. 5a. Note that among the drawings that contain the edge (v, u_2) the drawings of Figs. 5l and 5n are isomorphic, and the same holds for the drawings of Figs. 5m and 5o. Also, all obtained drawings are legal for the topological graph classes defined in the introduction, except for the class of 1-planar graphs.

in all possible ways (see Lines 6-13 of Algorithm 2). For this, we proceed mostly as above with one difference. Instead of half-pathways, we search for valid pathways for each edge (v, u_i), $2 \leq i \leq k$, i.e., we only consider pathways that start in a face incident to v and end in a face incident to u_i.

If we find an edge (v, u_i) for which no valid pathway exists, we declare that Γ cannot be extended to a simple drawing of G that respects the crossing restrictions of $\mathcal{C}$. Otherwise, the computed drawings of G are added to set $\mathcal{S}$, once all the drawings of $G \setminus \{v\}$ have been removed from it (see Lines 12 and 13 of Algorithm 2). To maintain our initial invariant, however, we remove from $\mathcal{S}$ drawings that are isomorphic to other drawings in $\mathcal{S}$ (see Line 8 of Algorithm 1).

Testing for isomorphism. We describe a procedure to test whether the planarizations Γ_1 and Γ_2 of two drawings of G comply with Properties P.1 and P.2 of a valid bijection.

We start by selecting two edges $e_1 = (v_1, w_1)$ and $e_2 = (v_2, w_2)$ in Γ_1 and Γ_2, respectively, whose end-vertices have compatible types (i.e., v_1 and v_2 are both real vertices or both crossings, and the same holds for w_1 and w_2). We bijectively map e_1 to e_2, v_1 to v_2, and w_1 to w_2, which complies with Property P.1. We call this a *base mapping* and try to extend it to a valid bijection.

Let f_1 be the face of Γ_1 that is "left" of e_1 (when walking along e_1 from v_1 to w_1). We bijectively map f_1 to one of the faces that are incident to e_2, which we call f_2. In the following we describe the procedure when f_2 is the face of Γ_2 that is "left" of e_2 (when walking along e_2 from v_2 to w_2). The case when f_2 is "right" of e_2 is symmetric. If the degrees of f_1 and f_2 are different, then the base mapping cannot be extended. Otherwise, both f_1 and f_2 have degree δ, and we walk simultaneously along their boundaries in counter-clockwise direction, starting at e_1 and e_2 respectively (when f_2 is "right" of e_2, we walk along the boundary of f_2 in clockwise direction). In view of Property P.2, for each $i = 1, \ldots, \delta$, we bijectively map the i-th vertex (either real or crossing) of f_1 to the i-th vertex of f_2, and the i-th edge of f_1 to the i-th edge of f_2. If a crossing is mapped to a real vertex, or if the degrees of two mapped vertices are different, then the base mapping cannot be extended.

If the vertices and edges of f_1 and f_2 have been mapped successfully, we proceed by considering the two maximal connected subdrawings Γ'_1 and Γ'_2 of Γ_1 and Γ_2, respectively, such that each edge of Γ'_1 and Γ'_2 has at least one face incident to it that is already mapped. Consider an edge e'_1 of Γ'_1 that is incident to only one mapped face f'_1 (such an edge exists, as long as the base mapping has not been completely extended). Let f''_1 be the other face incident to e'_1. Also, let e'_2 be the edge of Γ'_2 mapped to e'_1; note that e'_2 must be incident to a face f'_2 that is mapped to f'_1 and to a face f''_2 that is not mapped yet. We map to each other f''_1 and f''_2, and we proceed by applying the procedure described above (i.e., we walk along the boundaries of f''_1 and f''_2 simultaneously, while ensuring that the mapping remains valid). If this procedure can be performed successfully, then we have computed two subdrawings Γ''_1 and Γ''_2, such that $\Gamma'_1 \subseteq \Gamma''_1$, $\Gamma'_2 \subseteq \Gamma''_2$, and each edge of them has at least one face incident to it that is already mapped. Hence, we can recursively apply the aforementioned procedure to Γ_1'' and Γ_2''. This procedure eventually terminates since the number of faces of Γ_1 is bounded.

Drawings Γ_1 and Γ_2 are isomorphic, if the base mapping can be eventually extended. If this is not possible, then we have to consider another base mapping and check whether this can be extended. Note that the case where e_1 is bijectively mapped to e_2, v_1 to w_2, and w_1 to v_2 defines a different base mapping than the one we were currently considering. If none of the base mappings can be extended, then we consider Γ_1 and Γ_2 as non-isomorphic.

To reduce the number of base mappings that we have to consider, we first count the number of edges of Γ_1 and Γ_2 whose endpoints are both real vertices, both crossings, and those consisting of one real vertex and one crossing. These numbers have to be the same in Γ_1 and Γ_2. Since it is enough to consider base mappings only restricted to one of the three types of edges, we choose the type with the smallest positive number of occurrences. We summarize the above discussion in the following theorem.

Theorem 1 *Let G be a complete (or a complete bipartite) graph and let $\mathcal{C}$ be a beyond-planarity class of topological graphs. Then, G belongs to $\mathcal{C}$ if and only if, under the restrictions of class $\mathcal{C}$, our algorithm returns at least one drawing of G.*

4 Proof of Concept - Applications

In this section we use the algorithm described in Section 3 to test whether certain complete or complete bipartite graphs belong to specific beyond-planarity graph classes. We give corresponding characterizations and discuss how our findings are positioned within the literature (for an overview refer to Table 1). Our upper bounds are the smallest instances reported as negative by our algorithm. Our lower bound examples are drawings that certify membership to particular beyond-planarity graph classes, computed by an implementation of our algorithm; for typesetting reasons we have redrawn them. Our implementation is available to the community in the following repository:

`https://github.com/beyond-planarity/complete-graphs`

In the remainder of this section, we discuss our findings for different classes of graphs beyond planarity.

4.1 The class of k-planar graphs

In this section we consider k-planar graphs, in which each edge can be crossed at most k times. We start our discussion with the case of complete such graphs. As already mentioned in the introduction, the complete graph K_n is 1-planar if and only if $n \leq 6$ [28].

For the case of complete 2-planar graphs, the fact that a 2-planar graph with n vertices has at most $5n - 10$ edges [50] implies that K_9 is not a member of

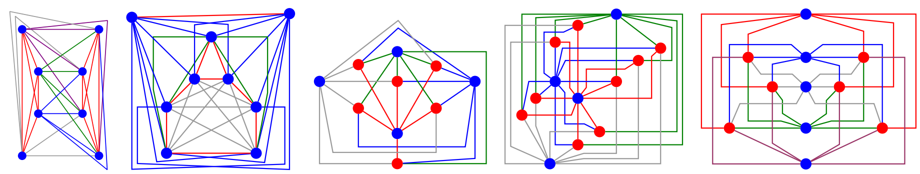


Figure 6: Illustration of (a) a 3-planar drawing of K_8, (b) a 4-planar drawing of K_9, (c) a drawing of $K_{4,6}$ that is both 2-planar and fan-crossing free, (d) a 3-planar drawing of $K_{4,9}$, and (e) a 3-planar drawing of $K_{5,6}$.

this class. Fig. 7 in [18], on the other hand, shows that K_7 is 2-planar. With our implementation we close this gap by reporting that even K_8 is not 2-planar.

For the cases of complete 3-, 4-, and 5-planar graphs, the application of a similar density argument as above proves that K_{10}, K_{11}, and K_{19} are not 3-, 4-, and 5-planar, respectively [2, 49]. With our implementation, we could conclude that even K_9 is not 3-planar, while K_{10} is neither 4- nor 5-planar. On the other hand, our algorithm was able to construct 3- and 4-planar drawings of K_8 and K_9, respectively; see Figs. 6a and 6b. Note that a 6-planar drawing of K_{10} can be easily derived from the 4-planar drawing of K_9 in Fig. 6b by adding one extra vertex inside the red colored triangle. The above results are summarized in the following characterization.

Characterization 2 *For $k \in \{1, 2, 3, 4\}$, the complete graph K_n is k-planar if and only if $n \leq 5 + k$. Also, K_n is 5-planar if and only if $n \leq 9$.*

Note that the 3-planarity of K_8 implies that the chromatic number of 3-planar graphs is lower bounded by 8. Analogous implications can be derived for the classes of 4-, 5-, and 6-planar graphs. Another observation that came out from our experiments is that, up to isomorphism, K_6 has a unique 1-planar drawing, K_7 has only two 2-planar drawings, and K_8 has only three 3-planar drawings, while the number of non-isomorphic 4-planar drawings of K_9 is significantly larger, namely 35. We provide more details about the numbers of non-isomorphic drawings in Section 5.

Consider now a complete bipartite graph $K_{a,b}$ with $a \leq b$. Note that $a \leq 2$ implies that $K_{a,b}$ is planar; thus, it trivially belongs to all beyond-planarity graph classes. Also, recall that $K_{a,b}$ is 1-planar if and only if $a \leq 2$, or $a = 3$ and $b \leq 6$, or $a = b = 4$ [28]. Further, a recent combinatorial result states that $K_{3,b}$ is k-planar if and only if $b \leq 4k + 2$ [9]. So, in the following we focus on the case where $a \geq 4$.

For complete bipartite 2-planar graphs, the fact that a bipartite 2-planar graph with n vertices has at most $3.5n - 7$ edges [10] implies that neither $K_{4,15}$ nor $K_{5,8}$ is 2-planar. With our implementation, we could conclude that $K_{4,7}$ and $K_{5,5}$ are not 2-planar, while $K_{4,6}$ is; we provide a corresponding certificate drawing in Fig. 6c. A summary of these results is given in the following characterization.

Characterization 3 *The complete bipartite graph $K_{a,b}$ (with $a \leq b$) is 2-planar if and only if (i) $a \leq 2$, or (ii) $a = 3$ and $b \leq 10$, or (iii) $a = 4$ and $b \leq 6$.*

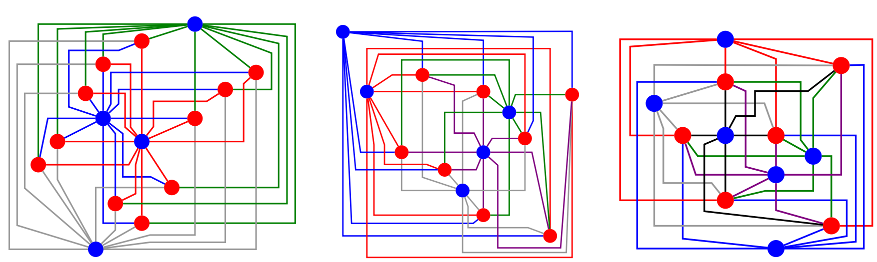
As opposed to the corresponding 2-planar case, there exists no upper bound on the edge density of 3-planar graphs tailored for the bipartite setting. The upper bound of $5.5n - 11$ edges [49] for general 3-planar graphs with n vertices does not provide any negative instance for $a \leq 5$, and only proves that $K_{6,b}$, with $b \geq 45$, is not 3-planar. With our implementation, we could provide significant improvements, by reporting that $K_{4,10}$, $K_{5,7}$, and $K_{6,6}$ are not 3-planar, while $K_{4,9}$ and $K_{5,6}$ are; we provide corresponding certificate drawings in Figs. 6d and 6e. Our results are summarized in the following characterization.

Characterization 4 *The complete bipartite graph $K_{a,b}$ (with $a \leq b$) is 3-planar if and only if (i) $a \leq 2$, or (ii) $a = 3$ and $b \leq 14$, or (iii) $a = 4$ and $b \leq 9$, or (iv) $a = 5$ and $b \leq 6$.*

On the other hand, we were unable to derive a complete picture for complete bipartite 4-planar graphs, but only some partial results, because the search space becomes drastically larger than in the previous cases and, as a consequence, our generation technique could not terminate. To give an intuition, note that $K_{4,4}$ has 81817 non-isomorphic 4-planar drawings, which makes the computation of the corresponding non-isomorphic drawings of $K_{4,5}$ infeasible in reasonable time. We provide more insights in Section 5.

On the positive side, we were able to report certificate drawings showing that $K_{4,11}$, $K_{5,8}$, and $K_{6,6}$ are 4-planar; see Fig. 7. We achieved this by slightly refining our generation technique. Namely, instead of computing *all* possible non-isomorphic simple drawings of graph $K_{a-1,b}$ or $K_{a,b-1}$, to compute the corresponding ones for $K_{a,b}$, we only computed few *samples*, in a *DFS-like* approach, aiming to eventually find a corresponding certificate drawing, only based on these samples. We summarize these findings in the following observation.

Observation 5 *The complete bipartite graph $K_{a,b}$ (with $a \leq b$) is 4-planar if (i) $a \leq 2$, or (ii) $a = 3$ and $b \leq 18$, or (iii) $a = 4$ and $b \leq 11$, or (iv) $a = 5$ and $b \leq 8$, or (v) $a = 6$ and $b = 6$. Further, $K_{a,b}$ is not 4-planar if $a \geq 3$ and $b \geq 19$.*



(a) (b) (c)

Figure 7: Illustration of 4-planar drawings of (a) $K_{4,11}$, (b) $K_{5,8}$ and (c) $K_{6,6}$.

4.2 The classes of fan-crossing and fan-planar graphs

In this section, we consider the classes of fan-crossing and fan-planar graphs. Recall that the former class does not allow an edge to be crossed by two independent edges, while the latter additionally does not allow an edge to be crossed by two adjacent edges from different directions. It is worth noting at this point that the class of fan-planar graphs is a proper subclass of the one of fan-crossing graphs [21], even though both classes have the same maximum edge density, namely, every n-vertex fan-crossing or fan-planar graph has at most $5n-10$ edges [21, 43]. Note that this bound is tight for both classes, as initially observed by Kaufmann and Ueckerdt [43]. In the following, we will notice that these two classes of graphs are "equivalent" also in terms of the largest complete and complete bipartite graphs belonging to them.

We start our discussion with complete graphs. The aforementioned density bound implies that K_9 is neither fan-crossing nor fan-planar, while Fig.7 in [18] shows that K_7 is fan-planar and thus fan-crossing. With our implementation, we can conclude that K_8 is not fan-crossing, and, as a consequence, not fan-planar. This yields the following characterization.

Characterization 6 *The complete graph K_n is fan-crossing or fan-planar if and only if $n \leq 7$.*

We note that Brandenburg in [23] claimed that the graph obtained from K_8 by removing one edge is not fan-crossing, but without giving the details of the proof of this claim. With a slight modification in our implementation, we could actually prove that the claim does not hold, since this graph is indeed fan-planar (and thus also fan-crossing); refer to Fig. 8 for an illustration.

Consider now a complete bipartite graph $K_{a,b}$ with $a \leq b$. For $a \leq 4$, Kaufmann and Ueckerdt [43] indicated that $K_{a,b}$ is fan-planar for any value of b, which implies that it is also fan-crossing. On the other hand, the fact that a bipartite fan-planar graph has at most $4n-12$ edges [10] implies that $K_{5,9}$ is not fan-planar (to the best of our knowledge, there exists no density bound for fan-crossing graphs that is tailored to bipartite graphs). Using our implementation, we concluded that even $K_{5,5}$ is not fan-crossing, and thus not fan-planar. These two results together imply the following characterization.

Characterization 7 *The complete bipartite graph $K_{a,b}$ (with $a \leq b$) is fan-crossing or fan-planar if and only if $a \leq 4$.*

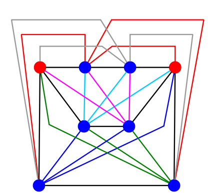


Figure 8: A fan-planar drawing of the graph obtained from K_8 by removing one edge, that is, the one connecting the two red colored vertices.

4.3 The class of fan-crossing free graphs

We continue our discussion with the class of fan-crossing free graphs, in which no edge can be crossed by two adjacent edges. A characterization for the case of complete graphs can be derived by combining two known results. First, K_6 is fan-crossing free, since it is 1-planar; with our implementation, we additionally demonstrate that, up to isomorphism, K_6 has a unique fan-crossing free drawing (see Section 5). Second, the fact that a fan-crossing free graph with n vertices has at most $4n - 8$ edges [27] implies that K_7 is not fan-crossing free. Hence, we have the following characterization.

Characterization 8 (Cheong et al. [27], Czap et al. [28]) *The complete graph K_n is fan-crossing free if and only if $n \leq 6$.*

As already stated, a combinatorial proof of the characterization of the complete bipartite fan-crossing free graphs is provided in the arXiv version [11] of this paper, where it is proved that $K_{4,6}$ is fan-crossing free, while $K_{3,7}$ and $K_{5,5}$ are not. We stress that the range of the case analysis in the proof is dramatically long. However, we could obtain the same result using our implementation.

Characterization 9 *The complete bipartite graph $K_{a,b}$ (with $a \leq b$) is fan-crossing free if and only if (i) $a \leq 2$, or (ii) $a \leq 4$ and $b \leq 6$.*

4.4 The class of gap-planar graphs

In this section, we continue our study with the class of gap-planar graphs, in which each crossing is assigned to one of its two involved edges, such that each edge can be assigned at most one crossing. A characterization of the complete gap-planar graphs has been recently provided by Bae et al. [15] as follows.

Characterization 10 (Bae et al. [15]) *The complete graph K_n is gap-planar if and only if $n \leq 8$.*

For the case of complete bipartite graphs, Bae et al. [15] proved that $K_{3,12}$, $K_{4,8}$, and $K_{5,6}$ are gap-planar, while $K_{3,15}$, $K_{4,11}$, and $K_{5,7}$ are not. These negative results were derived using the technique discussed in Section 1 that compares the crossing number of these graphs with their number of edges, which is an upper bound to the number of crossings allowed in a gap-planar drawing. By refining this technique, Bachmaier et al. [14] proved that even $K_{3,14}$, $K_{4,10}$, and $K_{6,6}$ are not gap-planar. Hence, towards a complete characterization one has to determine whether $K_{3,13}$ and $K_{4,9}$ are gap-planar or not. Here, we answer one of these two open questions by reporting that $K_{4,9}$ is in fact not gap-planar. Note that with our implementation we faced several difficulties in reporting whether $K_{3,13}$ is gap-planar or not, because of the number of non-isomorphic gap-planar drawings of $K_{3,7}$, which are more than 1,000,000 (up to the point of writing, after the program has been running for more than three months).

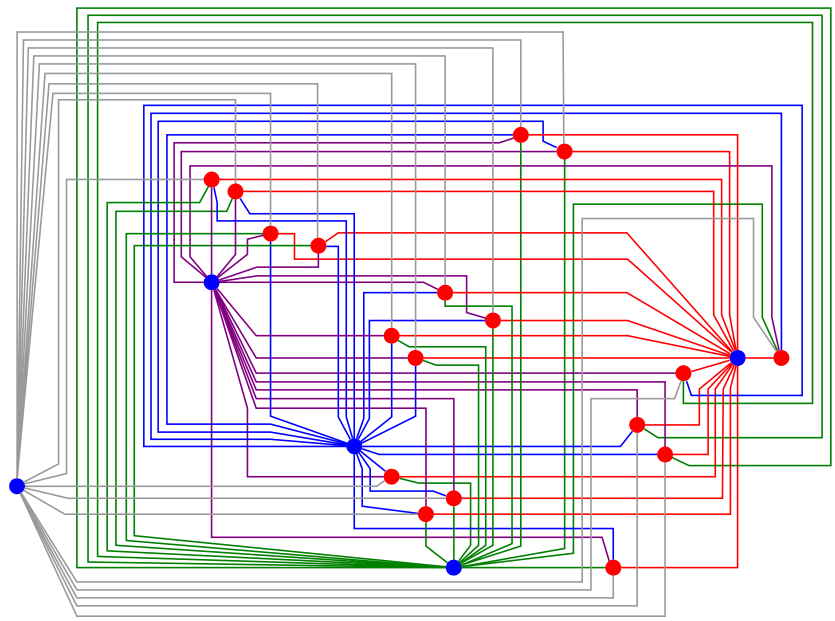


Figure 9: A quasiplanar drawing of $K_{5,18}$.

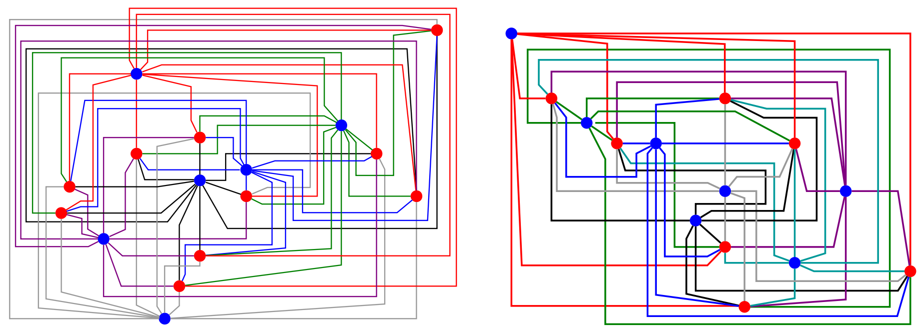
Observation 11 *The complete bipartite graph $K_{a,b}$ (with $a \leq b$) is gap-planar if (i) $a \leq 2$, or (ii) $a = 3$ and $b \leq 12$, or (iii) $a = 4$ and $b \leq 8$, or (iv) $a = 5$ and $b \leq 6$. Further, $K_{a,b}$ is not gap-planar if (i) $a = 3$ and $b \geq 14$, or (ii) $a = 4$ and $b \geq 9$, or (iii) $a = 5$ and $b \geq 7$, or (iv) $a \geq 6$ and $b \geq 6$.*

4.5 The class of quasiplanar graphs

In this section, we conclude our study with the class of quasiplanar graphs, which do not allow three mutually crossing edges. As in Section 4.4, a characterization for the complete quasiplanar graphs can be derived by combining two known results. First, the fact that a simple quasiplanar graph with n vertices has at most $6.5n - 20$ edges [4] implies that K_{11} is not quasiplanar. On the other hand, K_{10} is quasiplanar, as first observed by Brandenburg [20]. These two observations are summarized in the following characterization.

Characterization 12 (Ackerman et al. [4], Brandenburg [20]) *The complete graph K_n is quasiplanar if and only if $n \leq 10$.*

Consider now a complete bipartite graph $K_{a,b}$ with $a \leq b$. First, we observe that for $a \leq 4$, graph $K_{a,b}$ is quasiplanar for any value of b, since it is even fan-planar [43]. On the other hand, the fact that a quasiplanar graph with n vertices has at most $6.5n - 20$ edges [4] does not provide any negative answer for $a \leq 6$, while for $a = 7$ it only implies that $K_{7,52}$ is not quasiplanar. We stress that we were not able to find any improvement on the latter result. The reason is the



(a) (b)

Figure 10: Illustration of quasiplanar drawings of (a) $K_{6,10}$ and (b) $K_{7,7}$.

same as the one that we described for the class of complete bipartite 4-planar graphs (for further details, we point the reader to Section 5). Notably, using the DFS-like variant of our algorithm, we were able to derive at least positive certificate drawings for $K_{5,18}$, $K_{6,10}$, and $K_{7,7}$; see Figs. 9, 10a, and 10b. We summarize these findings in the following observation.

Observation 13 *The complete bipartite graph $K_{a,b}$ (with $a \leq b$) is quasiplanar if (i) $a \leq 4$, or (ii) $a = 5$ and $b \leq 18$, or (iii) $a = 6$ and $b \leq 10$, or (iv) $a = 7$ and $b \leq 7$. Further, $K_{a,b}$ is not quasiplanar if $a \geq 7$ and $b \geq 52$.*

5 Further insights from our implementation

In this section, we present some insights from the computations that we made in order to check whether certain complete and complete bipartite graphs belong to specific graph classes; for a summary refer to Table 2. Our algorithm was implemented in Java and was executed on a Windows machine with 2 cores at 2.9 GHz and 8 GB RAM.

As described in Section 3, our algorithm constructs all possible drawings of a certain (complete or complete bipartite) graph by adding a single vertex to the non-isomorphic drawings of the subgraph of it without this vertex. Once a new drawing is obtained in this procedure, we compare it for isomorphism against the already computed ones (and possibly discard it). The total number of produced drawings is reported in the column “General”, while the number of the non-isomorphic ones in the column “Non-Iso.”. The reported times are in seconds and correspond to the total time needed for generation and filtering for isomorphism. The bottommost row of each section in the table corresponds to a negative instance, as no drawing satisfying the constraints of the respective graph class could be found. The class of complete bipartite 4-planar graphs and the one of complete bipartite quasiplanar graphs form exceptions, as for these classes we were not able to report all non-isomorphic drawings of $K_{4,5}$.

Table 2: A summary of the required time (in sec.) and of the number of general and non-isomorphic drawings for different complete and complete bipartite graphs.

Part A: Results concerning the classes of k-planar graphs; $k \in \{1, 2, 3, 4\}$.

	complete				complete bipartite			
Class	Graph	General	Non-Iso.	Time	Graph	General	Non-Iso.	Time
1-planar	K_4	8	2	0.043	$K_{2,3}$	34	3	0.061
	K_5	13	1	0.043	$K_{3,3}$	14	2	0.049
	K_6	4	1	0.020	$K_{3,4}$	16	3	0.065
	K_7	0	0	0.006	$K_{4,4}$	5	2	0.044
					$K_{4,5}$	0	0	0.010
	total:	25	4	0.112	total:	69	10	0.229
2-planar	K_4	8	2	0.028	$K_{2,3}$	76	6	0.090
	K_5	89	4	0.105	$K_{3,3}$	243	19	0.254
	K_6	56	6	0.233	$K_{3,4}$	526	71	1.458
	K_7	38	2	0.119	$K_{4,4}$	310	38	1.152
	K_8	0	0	0.029	$K_{4,5}$	318	37	1.826
					$K_{5,5}$	0	0	0.357
	total:	191	14	0.514	total:	1473	171	5.137
3-planar	K_4	8	2	0.042	$K_{2,3}$	76	6	0.234
	K_5	109	5	0.195	$K_{3,3}$	678	69	1.802
	K_6	548	39	0.953	$K_{3,4}$	7141	1188	16.969
	K_7	648	39	3.459	$K_{4,4}$	24058	2704	97.801
	K_8	20	3	1.153	$K_{4,5}$	44822	7653	310.194
	K_9	0	0	0.065	$K_{5,5}$	20043	1899	199.908
					$K_{5,6}$	2516	438	47.396
					$K_{6,6}$	0	0	4.822
	total:	1333	88	5.867	total:	99334	13957	679.126
4-planar	K_4	8	2	0.040	$K_{2,3}$	76	6	0.108
	K_5	109	5	0.222	$K_{3,3}$	968	102	2.146
	K_6	1374	95	4.080	$K_{3,4}$	32454	6194	163.000
	K_7	14728	1266	79.842	$K_{4,4}$	681196	81817	34096.183
	K_8	7922	833	84.725	$K_{4,5}$	?	?	?
	K_9	353	35	33.672				
	K_{10}	0	0	1.175				
	total:	24494	2236	203.756	total:	?	?	?
5-planar	K_4	8	2	0.059				
	K_5	109	5	0.259				
	K_6	1752	119	4.716				
	K_7	83710	8318	1396.781				
	K_8	1190765	138750	262419.413				
	K_9	285847	29939	32299.196				
	K_{10}	0	0	2783.813				
	total:	1562191	177133	298904.237				

Part B: Results concerning the remaining graph classes considered in this paper.

	complete				complete bipartite			
Class	Graph	General	Non-Iso.	Time	Graph	General	Non-Iso.	Time
fan-crossing	K_4	8	2	0.034	$K_{2,3}$	76	6	0.110
	K_5	89	5	0.133	$K_{3,3}$	127	9	0.292
	K_6	147	39	0.226	$K_{3,4}$	295	43	0.757
	K_7	75	39	0.405	$K_{4,4}$	255	29	0.972
	K_8	0	0	0.196	$K_{4,5}$	324	48	1.624
					$K_{5,5}$	0	0	0.637
	total:	319	22	0.994	total:	1077	135	4.392
fan-crossing free	K_4	8	2	0.049	$K_{2,3}$	34	3	0.057
	K_5	13	1	0.054	$K_{3,3}$	38	5	0.092
	K_6	4	1	0.038	$K_{3,4}$	28	5	0.098
	K_7	0	0	0.009	$K_{4,4}$	19	4	0.106
					$K_{4,5}$	16	2	0.075
					$K_{5,5}$	0	0	0.012
	total:	25	4	0.150	total:	135	19	0.440
gap-planar	K_4	14	2	0.135	$K_{2,3}$	169	14	0.256
	K_5	243	10	0.366	$K_{3,3}$	1425	266	4.359
	K_6	739	237	4.726	$K_{3,4}$	16898	7466	170.396
	K_7	1124	665	13.943	$K_{3,5}$	148527	56843	12032.226
	K_8	1	1	16.347	$K_{4,5}$	199778	148367	28457.751
	K_9	0	0	0.019	$K_{4,6}$	408476	246318	132622.664
					$K_{4,7}$	173271	101428	32958.628
					$K_{4,8}$	5981	4015	2708.278
					$K_{4,9}$	0	0	99.583
	total:	2121	915	35.536	total:	954525	564717	209054.141
quasiplanar	K_4	8	2	0.082	$K_{2,3}$	76	6	0.187
	K_5	109	5	0.193	$K_{3,3}$	604	53	0.859
	K_6	936	63	1.820	$K_{3,4}$	11902	2248	34.073
	K_7	16505	1607	69.943	$K_{4,4}$	386241	46711	11328.401
	K_8	173199	20980	4044.264	$K_{4,5}$	?	?	?
	K_9	209248	23011	35163.772				
	K_{10}	81	9	7593.865				
	K_{11}	0	0	5.225				
	total:	400086	45677	46879.164	total:	?	?	?

As a typical example, we describe in the following one intermediate step in our computations; refer to the gray colored entry of Part A of Table 2. Our algorithm for reporting that $K_{6,6}$ is not a 3-planar graph generated at some intermediate step all 3-planar drawings of $K_{5,5}$, based on the non-isomorphic drawings of $K_{4,5}$. The algorithm reported in total 20043 drawings (including isomorphic ones), which were reduced to 1899 due to the elimination of isomorphic ones. These two steps together required 199.908 seconds. The obtained drawings were extended (by adding one additional vertex and its five incident edges) to 2516 drawings of $K_{5,6}$, which were reduced to 438 due to the filtering for isomorphism. None of these drawings could be extended to a 3-planar drawing of $K_{6,6}$, and thus we concluded that $K_{6,6}$ is not 3-planar.

The class of complete bipartite 4-planar graphs and the class of complete bipartite quasiplanar graphs show the limitations of our approach. We start our discussion with the former class. As already mentioned in Section 4.1, for the class of complete bipartite 4-planar graphs, we were able to report only some partial results (and not a complete characterization). The reason is depicted in Part A of Table 2. Observe that, in order to determine the 81817 non-isomorphic drawings of $K_{4,4}$, our implementation needed to generate 681196 drawings starting from the 6194 non-isomorphic drawings of $K_{3,4}$. This growth in the number of non-isomorphic drawings and the time needed to generate them (i.e., 34096 sec.) form a clear indication of the reason why our implementation failed to report all corresponding drawings of $K_{4,5}$. Similar observations can be made for the class of quasiplanar graphs; see Part B of Table 2.

We conclude this section by making some additional observations. First, it is eye-catching from both parts of Table 2 that the number of general and non-isomorphic drawings of the complete graphs are significantly smaller than the corresponding ones for the complete bipartite graphs. This observation is explained by the fact that the former are very symmetric and denser.

As it is naturally expected, we also observe that both the number of general drawings and the number of non-isomorphic drawings of a k-planar graph increases as k increases (at least for values of k in $\{1, 2, 3, 4, 5\}$). In particular, it seems that this increment becomes significantly large from 3- to 4-planar graphs, both in the complete and in the complete bipartite settings.

Comparing fan-crossing and fan-crossing free graphs, which are in a sense complementary to each other, we observe significant differences in the number of general and non-isomorphic drawings. In particular, the number of non-isomorphic drawings of fan-crossing free graphs are always single digits.

We finally observe that it is generally not a time-demanding task to conclude that a graph does not belong to a specific class, once all non-isomorphic drawings of its maximal realizable subgraph have been computed. In fact, the bottommost row of every section in Table 2 reports times in the order of few seconds at most.

Table 3: A comparison of the number of drawings reported by our algorithm with the elimination of isomorphic drawings (col. "General") and without it (col. "All") for the classes of 1- and 2-planar graphs; the corresponding execution times (in sec.) to compute these drawings are reported next to them.

	complete					complete bipartite				
Class	Graph	General	Time	All	Time	Graph	General	Time	All	Time
1-planar	K_4	8	0.043	8	0.043	$K_{2,3}$	34	0.061	34	0.061
	K_5	13	0.043	30	0.206	$K_{3,3}$	14	0.049	84	0.539
	K_6	4	0.020	120	0.737	$K_{3,4}$	16	0.065	960	5.642
	K_7	0	0.006	0	0.448	$K_{4,4}$	5	0.044	1584	10.871
						$K_{4,5}$	0	0.010	0	7.198
	total:	25	0.112	158	1.434	total:	69	0.229	2662	24.311
2-planar	K_4	8	0.028	8	0.028	$K_{2,3}$	76	0.090	76	0.090
	K_5	89	0.105	294	2.661	$K_{3,3}$	243	0.254	2352	10.571
	K_6	56	0.233	2664	3.292	$K_{3,4}$	526	1.458	52248	244.964
	K_7	38	0.119	8400	55.323	$K_{4,4}$	310	1.152	168624	1128.457
	K_8	0	0.029	0	51.321	$K_{4,5}$	318	1.826	1200384	8135.843
						$K_{5,5}$	0	0.357	0	12639.293
	total:	191	0.514	11366	112.625	total:	1333	5.137	1423684	22159.218

6 Conclusions and Open Problems

In this paper, we presented an efficient algorithm to generate all non-isomorphic drawings of complete (bipartite) graphs that are certificates of their membership to particular beyond-planarity graph classes. As a proof of concept, we obtained characterizations on the size of the largest such graphs for several classes. We remark that these results also have some theoretical implications. In particular, $K_{5,5}$ was conjectured in [10] not to be fan-planar; Characterization 7 implies that $K_{5,5}$ is not even fan-crossing, and thus settles in the positive this conjecture. By Characterization 7 and Observation 11, we deduce that $K_{5,5}$ is a certificate that there exist graphs which are gap-planar but not fan-planar. Since $K_{4,9}$ is fan-planar but not gap-planar, the two classes are incomparable, which answers a related question posed in [15] about the relationship between 1-gap-planar graphs and fan-planar graphs.

We stress that the elimination of isomorphic drawings is a key step in our algorithm, as shown in Table 3. For example, to test whether $K_{5,5}$ is 2-planar without the elimination of intermediate isomorphic drawings, one would need to investigate 1423684 drawings, while in the presence of this step only 1333. This significantly reduced the required time to roughly 5 seconds, including the time to perform all isomorphism tests and eliminations.

Our work leaves two main open problems. First, is it possible to extend our approach to graphs that are neither complete nor complete bipartite, e.g., to k-trees or to k-degenerate graphs (for small values of k)? A major difficulty is that, in the absence of symmetry, discarding isomorphic drawings becomes more complex. A general observation from our proof of concept is that our approach was of limited applicability on the classes of complete bipartite k-planar graphs, for $k > 3$, and complete bipartite quasiplanar graphs, for which we could report partial results. So, as a second open question, we ask whether it is possible to broaden these results by deriving improved upper bounds on the edge densities of these classes tailored for the bipartite setting (see, e.g., [10]).

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