Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 24, no. 1, pp. 21–46 (2020)
DOI: 10.7155/jgaa.00516

On the Circumference of Essentially 4-connected Planar Graphs

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Abstract

A planar graph is *essentially* 4-*connected* if it is 3-connected and every of its 3-separators is the neighborhood of a single vertex. Jackson and Wormald proved that every essentially 4-connected planar graph G on n vertices contains a cycle of length at least $\frac{2n+4}{5}$, and this result has recently been improved multiple times.

In this paper, we prove that every essentially 4-connected planar graph G on n vertices contains a cycle of length at least $\frac{5}{8}(n+2)$. This improves the previously best-known lower bound $\frac{3}{5}(n+2)$.

Submitted:	Reviewed:	Revised:	Accepted:
February 2019	August 2019	September 2019	December 2019
	Final:	Published:	
	December 2019	January 2020	

Article type:	Communicated by:
Regular paper	G. Liotta

Research partially supported by DAAD, Germany (as part of BMBF) and by the Ministry of Education, Science, Research and Sport of the Slovak Republic within the project 57320575. IF is also partially supported by Science and Technology Assistance Agency under the contract No. APVV-15-0116 and by the Slovak VEGA Grant 1/0368/16. SM and JMS are also partially supported by the grants 327533333 and SCHM 3186/1-1 (270450205) from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation). Gefördert durch die Deutsche Forschungsgemeinschaft (DFG) – 327533333 und 270450205.

1 Introduction

The *circumference* circ(G) of a graph G is the length of a longest cycle of G. Originally being the subject of Hamiltonicity studies, essentially 4-connected planar graphs and their circumference have been thoroughly investigated throughout literature. Jackson and Wormald [5] proved that circ$(G) \geq \frac{2n+4}{5}$ for every essentially 4-connected planar graph G on n vertices. An upper bound is given by an infinite family of essentially 4-connected planar graphs G such that circ$(G) = \frac{2}{3}(n+4)$ [2]. Fabrici, Harant and Jendroľ [2] improved recently the lower bound to circ$(G) \geq \frac{1}{2}(n+4)$; this result in turn was strengthened to circ$(G) \geq \frac{3}{5}(n+2)$ in [3]. It remained an open problem whether every essentially 4-connected planar graph G on n vertices satisfies circ$(G) > \frac{3}{5}(n+2)$.

In this paper, we present the following result.

Theorem 1 *Every essentially 4-connected planar graph G on n vertices contains a cycle of length at least $\frac{5}{8}(n+2)$. If $n \geq 16$, circ$(G) \geq \frac{5}{8}(n+4)$.*

This result encompasses most of the results known for the circumference of essentially 4-connected planar graphs (some of which can be found in [2, 4, 8]). In particular, it improves the bound circ$(G) \geq \frac{13}{21}(n+4)$ that has been given in [2] for the special case that G is maximal planar for sufficiently large n (in fact, for every $n \geq 16$, as explained in Section 4).

2 Preliminaries

Throughout this paper, all graphs are simple, undirected and finite. For a vertex x of a graph G, denote by $\deg_G(x)$ the degree of x in G. For a vertex subset $A \subseteq V$, let the *neighborhood* $N_G(A)$ of A consist of all vertices in $V - A$ that are adjacent to some vertex of A. For vertices $v_1, v_2, \ldots, v_i$ of a graph G, let $(v_1, v_2, \ldots, v_i)$ be the path of G that visits the vertices in the given order. We omit subscripts if the graph G is clear from the context.

A *separator* S of a graph G is a subset of V such that $G - S$ is disconnected; S is a *k-separator* if $|S| = k$. A separator S is *trivial* if at least one component of $G - S$ is a single vertex, and *non-trivial* otherwise. Let a graph G be *essentially 4-connected* if G is 3-connected and every 3-separator of G is trivial. It is well-known that, for every 3-separator S of a 3-connected planar graph G, $G - S$ has exactly two components.

A cycle C of a graph G is *isolating* (sometimes also called *outer-independent*) if every component of $G - V(C)$ is a single vertex that has degree three in G. An edge xy of a cycle C of G is *extendable* if x and y have a common neighbor in $G - V(C)$. For example, Figure 2 depicts (a part of) an isolating cycle C for which the edge yz becomes extendable after contracting the edge zu. According to Whitney [7], every 3-connected planar graph has a unique embedding into the plane (up to flipping and the choice of the outer face). Hence, we assume in the following that the embeddings of such graphs are fixed.

3 Proof of Theorem 1

Let G be an essentially 4-connected plane graph. It is well-known that every 3-connected plane graph on at most 10 vertices is Hamiltonian [1]; thus, for $4 \leq n \leq 10$, this implies $\mathrm{circ}(G) = n \geq \frac{5}{8}(n+2)$. Since these graphs contain in particular the essentially 4-connected plane graphs on at most 10 vertices, we assume $n \geq 11$ from now on. For $n \geq 11$, it was shown in [2, Lemma 4(ii)] that G contains an isolating cycle of length at least 8. Let C be a longest such isolating cycle of length $c := |E(C)| \geq 8$. We will show that $c \geq \frac{5}{8}(n+2)$, so that C is a cycle of the desired length.

Clearly, C contains no extendable edge xy, as otherwise one could find a longer such cycle by replacing xy in C with the path (x, v, y), where $v \notin V(C)$ is a common neighbor of x and y. Let V^- be the subset of vertices of V that are contained in the open set of $\mathbb{R}^2 - C$ that is bounded (hence, strictly inside C), and let $V^+ := V - V(C) - V^-$. We assume that $|V^-| \geq 1 \leq |V^+|$, since otherwise we are done, as then $c \geq \frac{2}{3}(n+2)$ is implied by [2, Lemma 5]. Let H be the plane graph obtained from G by deleting all chords of C (i. e., all edges $xy \in E - E(C)$ satisfying $x, y \in V(C)$) and let $H^- := H - V^+$ and $H^+ := H - V^-$. A face of H is called *minor* if it is incident to exactly one vertex of $V^- \cup V^+$, and *major* otherwise. Let M^- and M^+ be the sets of minor faces in H^- and H^+, respectively. For example, in Figure 2, we have $a \in V^-$, $b \in V^+$, $f \in M^-$ and $f' \in M^+$.

Note that a face f of H is incident to no vertex of $V^- \cup V^+$ if and only if it is bounded by C (i.e., if f is either the region inside or outside C). Since we assumed $|V^-| \geq 1 \leq |V^+|$, our definition of minor faces coincides with the one of [3], so that we can use the following inequality.

Lemma 1 ([3], Inequality (i)) $|M^- \cup M^+| \geq |V^- \cup V^+| + 2$.

In H, an edge e of C is incident with exactly two faces f and f' of H. In this case we say f' is *opposite* to f with respect to e. A face f of H is called *j-face* if it is incident with exactly j edges of C; the edges of C that are incident with f are called *C-edges* of f. Since C does not contain an extendable edge, we have $j \geq 2$ for every minor j-face of H. For two faces f and f' of H, let $m_{f,f'}$ be the number of common C-edges of f and f'.

If we can prove

$$2c \geq \frac{10}{3}|M^- \cup M^+|, \tag{1}$$

then Theorem 1 follows directly from the inequality $|M^- \cup M^+| \geq n - c + 2$ of Lemma 1. We charge every j-face of H with weight j (and thus have a total charge of weight $2c$) and discharge these weights in H by applying the following set of rules exactly once. In order to prove Inequality (1), we will aim to prove that every minor face of H has weight at least $^{10}/_3$ after the discharging.

Rule R1: Every major face f of H sends weight $m_{f,f'}$ to every minor face f' opposite to f.

Rule R2: Every minor face f of H sends weight $\frac{2}{3}m_{f,f'}$ to every minor 2-face f' opposite to f.

Rule R3: Every minor face f of H sends weight 1 to every minor 3-face f' that is opposite to f with respect to the middle C-edge of f'.

Rule R4: Let f_1 be a minor 4-face that has an opposite minor j-face f satisfying $j \geq 4$ and $m_{f_1,f} = 2$, as well as an opposite minor 2- or 3-face f_2 satisfying $m_{f_1,f_2} = 2$. Then f sends weight $2/3$ to f_1.

Rule R5: Let f_1 be a minor 5-face that has an opposite minor j-face f satisfying $j \geq 4$ and $m_{f_1,f} = 2$, as well as two opposite minor 2-faces. Then f sends weight $1/3$ to f_1.

For example, in Figure 2, both faces f and f' would send weight 2/3 to each other according to Rule R2, which effectively cancels the exchange of weights. Rules R2 and R3 may be seen as a refinement of the two rules given in [3]; for that reason, some of the early cases about minor 2- and 3-faces in the following case distinction will be similar as in [3].

Let w denote the weight function on the set $F(H)$ of faces of H after Rules R1–R5 have been applied. Clearly, $\sum_{f \in F(H)} w(f) = 2c$ still holds. In order to prove that the weight $w(f)$ of every minor face f of H is at least $10/3$ and no major face has negative weight, we distinguish several cases. For most of them, we construct a cycle $\overline{C}$ that is obtained from C by replacing a subpath of C with another path. In such cases, $\overline{C}$ will be an isolating (which is easy to verify due to $V(C) \subseteq V(\overline{C})$) cycle of G that is longer than C (we say C is *extended*); this contradicts the choice of C and therefore shows that the considered case cannot occur. Note that the vertices of C that are depicted in the following figures are pairwise non-identical, because $c \geq 8$; in the rare figures that show more than 8 vertices of C, C has always at least the number of vertices shown.

Let $f \in F(H)$**.**

Case 1: *f is a major j-face for any j.*

Initially, f is charged with weight j. By Rule R1, f sends for every of its C-edges weight at most 1 to an opposite face. We conclude $w(f) \geq 0$.

Case 2: *f is a minor 2-face* (see Figure 1).

Let xy and yz be the C-edges of f and let a be the vertex of $V - V(C)$ that is incident with f. The face f is initially charged with weight 2 and gains weight at least $4/3$ by R1 and R2. If f does not send any weight to other faces, this gives $w(f) \geq 10/3$, so assume that f sends weight to some face $f' \neq f$.

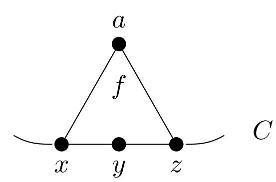

Figure 1: Case 2

According to R1–R5, f' is opposite to f and either a minor 2-face or a minor 3-face of H. Without loss of generality, let f' be opposite to f with respect to the edge yz. We distinguish the following subcases.

Case 2a: *f' is a minor 2-face and xy is a C-edge of f'.*

Then $\{x, z\}$ is the neighborhood of y in G, which contradicts the 3-connectivity of G.

Case 2b: *f' is a minor 2-face and xy is not a C-edge of f'* (see Figure 2).

Then a longer isolating cycle $\overline{C}$ is obtained from C by replacing the path (x, y, z, u) with the path (x, a, z, y, b, u) (see Figure 2), which contradicts the choice of C.

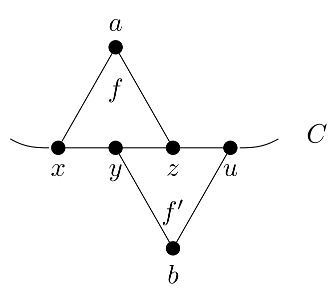

Figure 2: Case 2b

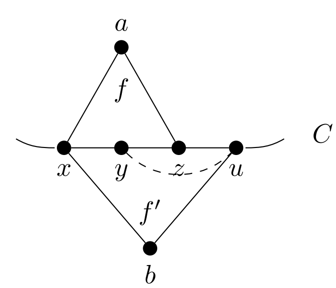

Figure 3: Case 2c

Case 2c: *f' is a minor 3-face* (see Figure 3).

Since we assumed that f sends weight to f', one C-edge of f, say without loss of generality yz, is the middle C-edge of f', according to R3. The edge yu (see Figure 3) exists in G (but not in H, as H does not contain chords of C), because otherwise $d_G(y) = 2$, which contradicts that G is 3-connected. Then $\overline{C}$ is obtained from C by replacing the path (x, y, z, u) with the path (x, a, z, y, u).

Case 3: *f is a minor 3-face* (see Figure 4).

Then f is initially charged with weight 3 and gains weight at least 1 by R1 and R3. If f sends weight at most $2/3$ to other faces, this gives

$w(f) \geq 10/3$, so assume that f sends weight more than $2/3$. Since all weights are multiples of $1/3$, f has to send weight at least $3/3$. In particular, this implies that Rule R2 or R3 applies on f.

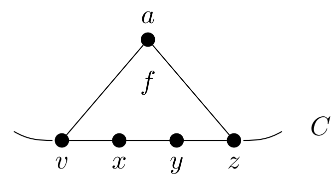

Figure 4: Case 3

Let f_1, f_2 and f_3 be the (possibly identical) opposite faces of f with respect to the C-edges vx, xy, yz of f (see Figure 4). Then f_2 is not a minor 2-face for the same reason as in Case 2c. We distinguish the following subcases.

Case 3a: *Neither* f_1 *nor* f_3 *is a minor* 3*-face* (see Figure 5).

Then f_2 is neither a minor 2-face nor a minor 3-face, and f_1 and f_3 are minor 2-faces, as otherwise by R1–R5 f would not send a total weight of more than $2/3$ to its opposite faces. Moreover, $b \neq d$ (in the notation of Figure 5), since xy is not extendable. Then $\overline{C}$ is obtained from C by replacing the path (w, v, x, y, z, u) with the path $(w, b, x, v, a, z, y, d, u)$.

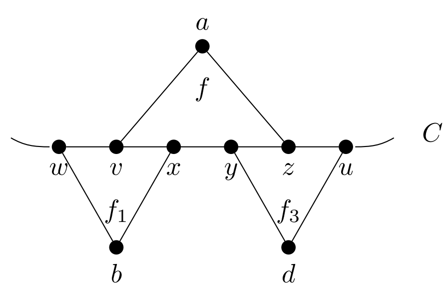

Figure 5: Case 3a

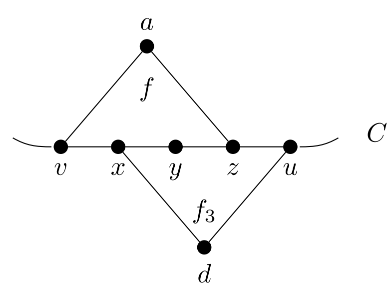

Figure 6: Case 3b

Case 3b: f_1 *or* f_3 *is a minor* 3*-face* (see Figure 6).

The face f_2 is not a minor 3-face with middle C-edge xy, as otherwise $\{v, z\}$ would be a 2-separator of G. Hence, $f_1 \neq f_3$. Since f sends a total weight of more than $2/3$ to its opposite faces, at least one of f_1 and f_3 is a minor 3-face that has its middle C-edge in $\{vx, yz\}$ by R3, say without loss of generality that the middle C-edge of f_3 is yz. Then $\overline{C}$ is obtained from C by replacing the path (v, x, y, z, u) with the path (v, a, z, y, x, d, u).

Case 4: *f is a minor 4-face* (see Figure 7).

Then f is initially charged with weight 4. If f looses a total net weight of at most $2/3$, then $w(f) \geq 10/3$, so assume that weight at least $3/3$ is sent to opposite faces. We have to show that this is impossible by considering Rules R2–R5.

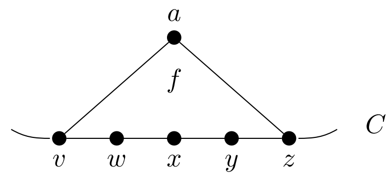

Figure 7: Case 4

Assume first that f has an opposite minor 2-face f'. We distinguish the following subcases.

Case 4a: *f' has C-edges wx and xy* (see Figure 8).

Then vx or xz is an edge of G and C can be extended by detouring C through one of these edges and d, which contradicts the choice of C.

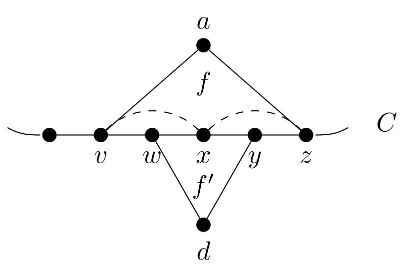

Figure 8: Case 4a

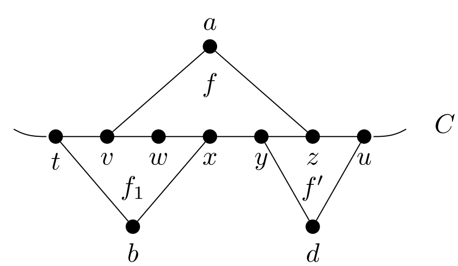

Figure 9: Case 4b

Case 4b: *Every opposite minor 2-face of f has exactly one C-edge of f* (see Figure 9).

In particular, $m_{f,f'} = 1$. Without loss of generality, let f' have the C-edge yz. Then f sends weight $2/3$ to f' by R2, and R1 does not decrease the weight of f. Moreover, if f sends weight to another face with the Rules R4 or R5, then xy is a C-edge of a major face (since C does not contain any extendable edge) and f gains weight 1 from this major face, so that $w(f) \geq 4 - 2/3 + 1 - 2/3 = 11/3$, which contradicts $w(f) < 10/3$. Therefore, f has by R2 and R3 an opposite minor 2- or 3-face $f_1 \neq f'$. If f_1 is a minor 2-face, $m_{f,f_1} = 1$, so that f_1 has the C-edge vw. Then neither wx nor xy is a C-edge of a minor face opposite to f, as such a minor face would be a 2-face with C-edges wx and xy (see Case 4a). Thus, f gains weight 2 from the major face(s) with C-edges wx and xy, which contradicts $w(f) < 10/3$.

Hence, f_1 is a minor 3-face. Since $w(f) < 10/3$, the middle C-edge of f_1 is either vw or wx. If it is vw, $\overline{C}$ can be obtained from C by replacing the path (t, v, w, x, y, z, u) with $(t, b, x, w, v, a, z, y, d, u)$ (see Figure 9), as we have $b \neq d$, since otherwise C would contain the extendable edge xy. Hence, let the middle C-edge of f_1 be wx. Then $wz \notin E(G)$, as otherwise C could be extended by replacing the path (v, w, x, y, z) with (v, b, y, x, w, z). Since $\{v, y\}$ is not a 2-separator of the 3-connected graph G, this implies $xz \in E(G)$. Then $\overline{C}$ can be obtained from C by replacing the path (x, y, z, u) with (x, z, y, d, u), which contradicts the choice of C.

From Cases 4a+b, we conclude that f' has either the C-edges vw and wx or the C-edges xy and yz, say without loss of generality the latter.

Case 4c: *f' has C-edges xy and yz, and f has an opposite major face* (see Figure 10).

Then $wy \notin E(G)$, as otherwise C can be extended by detouring through f'. Hence, $vy \in E(G)$, as otherwise $\deg_G(y) = 2$. Since f has an opposite major face and wx is not an extendable edge of C, wx is a C-edge of such an opposite major face f''. Then f gains weight 1 from f'' by R1 and sends by R2 weight $2/3$ to a minor opposite 2-face with C-edge vw in order to satisfy the assumption $w(f) < 10/3$ (see Figure 10 and note that R4 and R5 do not apply here). But this is impossible, as then C can be extended by replacing the path (t, v, w, x, y, z) with (t, b, w, v, y, x, d, z), since $b \neq d$.

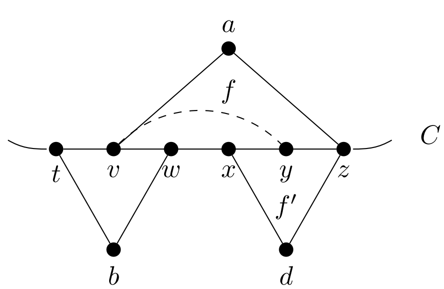

Figure 10: Case 4c

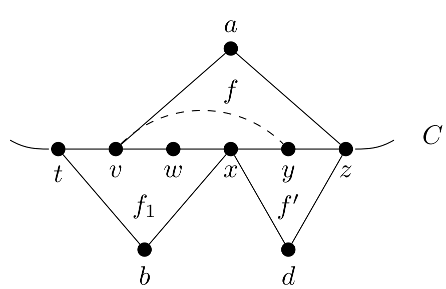

Figure 11: Case 4d

Case 4d: *f' has C-edges xy and yz, and wx is a C-edge of a minor 2- or 3-face f_1* (see Figure 11).

As in Case 4c, $wy \notin E(G)$ and $vy \in E(G)$. Hence, f_1 is a minor 3-face, as otherwise $\deg_G(w) = 2$. Then $\overline{C}$ is obtained from C by replacing the path (t, v, w, x, y, z) with (t, b, x, w, v, y, z) (note that $b = d$ is possible).

Case 4e: *f' has C-edges xy and yz, and wx is a C-edge of a minor j-face f_1 with $j \geq 4$* (see Figure 12).

Then f gains weight $2/3$ from f_1 by R4 and sends weight $4/3$ to f'. Hence, we get the contradiction $w(f) = 10/3$, unless f sends weight $2/3$ to f_1 by R4 or $1/3$ to f_1 by R5. In that case, $j = 4$ or $j = 5$ and there are only minor 2-faces opposite to f_1. As argued in Case 4c, $wy \notin E(G)$ and $vy \in E(G)$. Moreover, uw (and su in case of $j = 5$; see Figure 12) are not edges of G, as otherwise C can be extended by detouring through g. Hence, $ux \in E(G)$, as otherwise $\deg_G(u) = 2$, which is a contradiction. This implies $\deg_G(w) = 2$, which is a contradiction.

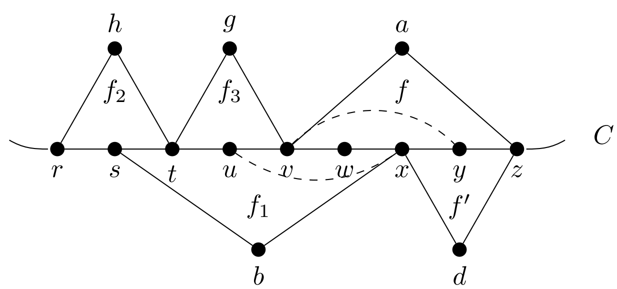

Figure 12: Case 4e

From Cases 4a–e, we conclude that f has no opposite minor 2-face. Then $w(f) < 10/3$ and R1–R5 imply that f has an opposite minor 3-face that has a C-edge of f as middle C-edge (due to R3), or an opposite minor 4-face f' with $m_{f,f'} = 2$ that has an opposite minor 2- or 3-face f_2 with $m_{f',f_2} = 2$ (due to R4); note that we still contradict $w(f) < 10/3$ when f has two opposite minor 5-faces, to each of which f sends weight $1/3$ by R5. We therefore distinguish these remaining subcases.

Case 4f: *f has an opposite minor 3-face f' with middle C-edge wx or xy* (see Figure 13).

Without loss of generality, let xy be the middle C-edge of f'. Then $vy \notin E(G)$, as otherwise C can be extended by replacing the path (v, w, x, y, z) with (v, y, x, w, d, z). This implies $wy \in E(G)$, as otherwise $\deg_G(y) = 2$. Since $\{w, z\}$ is no 2-separator of G, $vx \in E(G)$. Then C can be extended by replacing the path (v, w, x, y, z) with (v, x, y, w, d, z).

Case 4g: *f has an opposite minor 3-face f' with middle C-edge vw or yz, but no opposite 4-face* (see Figure 14).

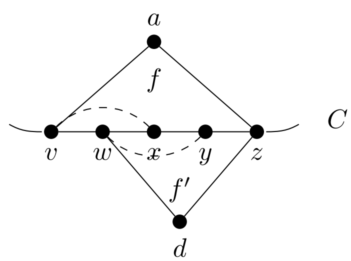

Figure 13: Case 4f

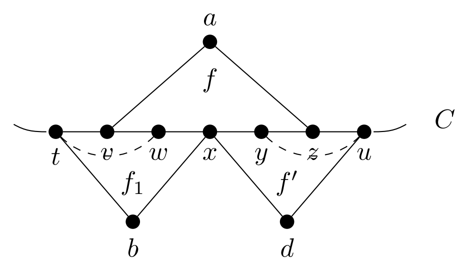

Figure 14: Case 4g

Without loss of generality, let yz be the middle C-edge of f'. Let f_1 be the face opposite to f that has C-edge wx. Then f_1 is not major, as otherwise $w(f) = 4 - 1 + 1 > 10/3$, since f has no opposite minor 2-faces. For the same reason, f_1 is a minor j-face satisfying $j \geq 3$. If $j \geq 5$, f_1 sends weight $2/3$ to f due to R4, which contradicts $w(f) < 10/3$, as f sends weight at most $1/3$ to f_1 due to R5 (exactly $1/3$ only if $j = 5$ and f_1 has two opposite 2-faces).

Since $j \neq 4$ by assumption, f_1 is a minor 3-face (see Figure 14). Then $wy \notin E(G)$, as otherwise $\overline{C}$ is obtained from C by replacing the path (v, w, x, y, z, u) with (v, a, z, y, w, x, d, u), and $wz \notin E(G)$, as otherwise $\overline{C}$ is obtained from C by replacing the path (w, x, y, z, u) with (w, z, y, x, d, u). Hence, $tw \in E(G)$, as otherwise $\deg_G(w) = 2$. Then $\overline{C}$ is obtained from C by replacing the path (t, v, w, x, y, z, u) with $(t, w, v, a, z, y, x, d, u)$, which contradicts the choice of C.

Case 4h: *f has an opposite minor 3-face f' with middle C-edge vw or yz and an opposite 4-face f_1* (see Figure 15).

Without loss of generality, let yz be the middle C-edge of f'. Then $m_{f,f_1} = 2$, as otherwise wx is a C-edge of a major face, which would imply $w(f) = 4 - 1 + 1 > 10/3$. Hence, f_1 sends weight $2/3$ to f by R4, which implies that f must send weight $2/3$ to f_1 by R4, as otherwise $w(f) \geq 10/3$. Hence, f_1 has an opposite minor 2- or 3-face f_2 that satisfies $m_{f_1,f_2} = 2$ (see Figure 15). Then $wy \notin E(G)$, as otherwise C can be extended by replacing the path (v, w, x, y, z, q) with (v, a, z, y, w, x, d, q), and $wz \notin E(G)$, as otherwise C can be extended by replacing the path (w, x, y, z, q) with (w, z, y, x, d, q). If f_2 is a 3-face, this implies by symmetry $tw \notin E(G)$ and $uw \notin E(G)$, which contradicts $\deg_G(w) \geq 3$. Hence, f_2 is a 2-face. Then $uw \notin E(G)$, as otherwise C can be extended by replacing the path (t, u, v, w) with (t, g, v, u, w), which implies $tw \in E(G)$, as otherwise $\deg_G(w) = 2$. This contradicts $\deg_G(u) \geq 3$.

Case 4i: *f has no opposite minor 3-face whose middle C-edge is a C-edge of f* (see Figure 16).

Then, as argued before, f has an opposite minor 4-face f' with

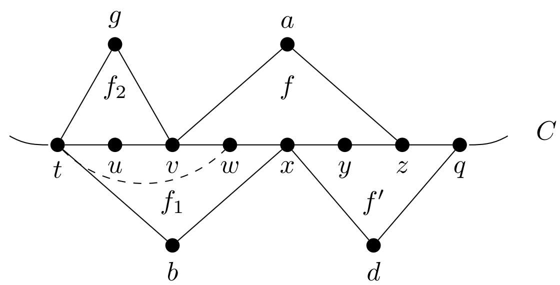

Figure 15: Case 4h

$m_{f,f'} = 2$ and C-edges xy and yz, that has an opposite minor 2- or 3-face f_2 with $m_{f',f_2} = 2$. According to R4, f sends weight $2/3$ to f'. Let f'' be the face opposite to f that has C-edge wx. Then f'' must be either a second opposite minor 4-face with $m_{f,f''} = 2$ that has an opposite minor 2- or 3-face f_1 with $m_{f'',f_1} = 2$ (due to R4), or a opposite minor 5-face with $m_{f,f''} = 2$ that has two opposite minor 2-faces (due to R5), as otherwise $w(f) \geq 4 - 2/3 = 10/3$, since f sends no weight to any 2- or 3-face by R2 or R3. Note that $g = a = h$ and $b = d$ are possible.

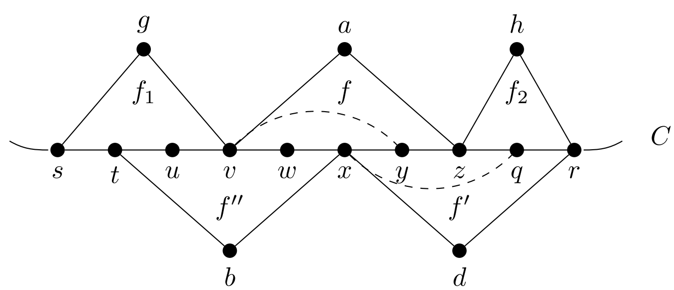

Figure 16: Case 4i

We claim that in all cases vy is an edge of G. Consider the case that f_2 is a 2-face (see Figure 16). Then $yq \notin E(G)$, as otherwise C can be extended by replacing the path (y, z, q, r) with (y, q, z, h, r), and thus $xq \in E(G)$, as otherwise $\deg_G(q) = 2$. This implies that vy or wy is in G, as otherwise $\deg_G(y) = 2$. Since $wy \notin E(G)$, as otherwise C can be extended by replacing the path (w, x, y, z, q, r) with (w, y, x, q, z, h, r), we have $vy \in E(G)$, as claimed. Now consider the remaining case that f_2 is a 3-face. By symmetry, we will assume instead that f_1 is a 3-face and prove that $wz \in E(G)$ (such that the notation of Figure 16 can be used); this implies $vy \in E(G)$ for the case that f_2 is a 3-face. Then $wy \notin E(G)$, as otherwise C can be extended by replacing the path (s, t, u, v, w, x, y) with $(s, g, v, u, t, b, x, w, y)$, and $uw \notin E(G)$, as otherwise C can be extended by replacing the path (s, t, u, v, w, x) with (s, g, v, w, u, t, b, x). In addition, $tw \notin E(G)$, as otherwise C can be extended by replacing the path (s, t, u, v, w) with (s, g, v, u, t, w). Then $wz \in E(G)$, as claimed, since otherwise $\deg_G(w) = 2$, which is a contradiction.

Hence, we proved that in all cases $vy \in E(G)$. If f'' is a 5-face, then $ux \in E(G)$ by the last argument of Case 4e, which contradicts $\deg_G(w) \geq 3$. Hence, f'' is a 4-face, and no matter whether f_1 is a 2- or 3-face, wz is an edge of G by a symmetric argument to the one of the last paragraph. This contradicts that G is plane, because $vy \in E(G)$.

Case 5: *f is a minor 5-face* (see Figure 17).

Then f is initially charged with weight 5. If f looses a total net weight of at most $5/3$, then $w(f) \geq 10/3$, so assume otherwise. We distinguish the following subcases.

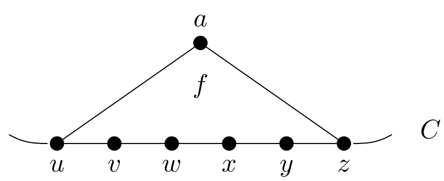

Figure 17: Case 5

Case 5a: *f sends weight to an opposite minor 5-face f'* (see Figure 18).

Without loss of generality, let xy and yz be C-edges of f' by R5. Then f sends weight $1/3$ to f', and f' has two opposite minor 2-faces f_1 and f_2. Since $w(f) < 10/3$, f does neither send weight to a second 5-face nor to a 4-face nor to a 3-face (as there may be at most one of each kind and, if so, no 2-face that receives weight from f). This implies that the edge uv is a C-edge of a minor 2-face f_3 opposite to f, and that vw and wx are the C-edges of a second minor 2-face f_4 opposite to f (see Figure 18). Then f' sends weight $1/3$ back to f by R5, but $w(f) = 5 - 3 \cdot \frac{2}{3} = 3 < 10/3$ is still satisfied.

We have $yp \notin E(G)$ and $pr \notin E(G)$, as otherwise C can be extended by detouring through g. Since $\deg_G(p) \geq 3$, $xp \in E(G)$. By symmetry, $wz \in E(G)$, which implies $yw \in E(G)$. Then C can be extended by replacing the path (v, w, x, y) with (v, b, x, w, y).

Case 5b: *f sends weight to an opposite minor 4-face f'* (see Figure 19).

Without loss of generality, let xy and yz be C-edges of f' by R4. Assume first that f sends weight to an opposite minor 3-face f_1.

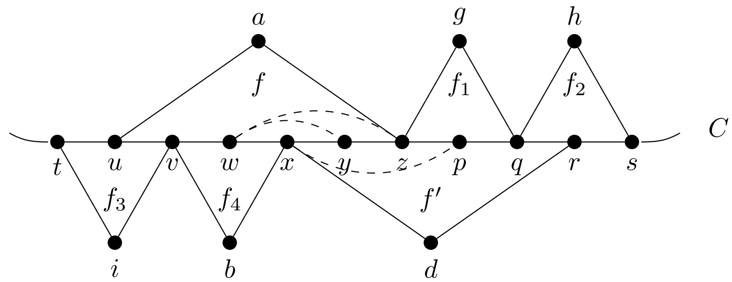

Figure 18: Case 5a

Then f sends total weight $5/3$ to f' and f_1, and the middle C-edge of f_1 is either uv or vw. Both cases contradict $w(f) < 10/3$, since no further weight is sent. The same argument gives a contradiction if f sends weight to a minor 4-face different from f'.

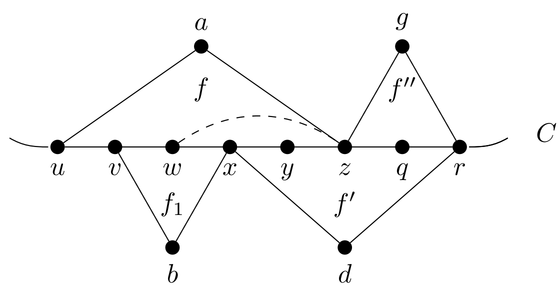

Figure 19: Case 5b

Hence, f sends a total weight of at least $4/3$ to minor 2-faces, as R2 sends only multiples of weight $2/3$. This implies that f has an opposite minor 2-face f_1 with $m_{f,f_1} = 2$. If f_1 has C-edges uv and vw, then wx is again a C-edge of major face, which sends weight 1 to f and thus contradicts $w(f) < 10/3$. Hence, f_1 has C-edges vw and wx (see Figure 19). Then uw and wy are not edges of G, as otherwise C can be extended by detouring through b. Hence, $wz \in E(G)$, as otherwise $\deg_G(w) = 2$. Moreover, $yq \notin E(G)$ and $xq \in E(G)$ for the same reason as in Case 4i, which contradicts $\deg_G(y) \geq 3$.

Case 5c: *f sends weight to an opposite minor 3-face f' with middle C-edge wx* (see Figure 20).

In order to have $w(f) < 10/3$, by R1–R3, f sends weight $2/3$ to each of the minor 2-faces f_1 and f_2 having C-edges uv and yz, respectively. Then uw and xz are not edges of G, as otherwise C can be extended by detouring C through b or g, respectively. Since $\{v, y\}$ is not a 2-

separator of G, this implies that either $wz \in E(G)$ or $ux \in E(G)$, say by symmetry the former. Then we can obtain $\overline{C}$ from C by replacing the path (v, w, x, y, z) with (v, d, y, x, w, z).

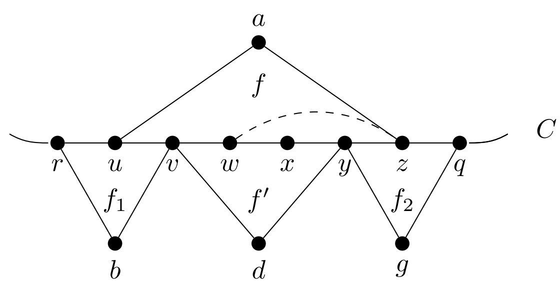

Figure 20: Case 5c

Case 5d: *f sends weight to an opposite minor 3-face f' with middle C-edge vw or xy, but not to any opposite minor 4- or 5-face* (see Figure 21).

Without loss of generality, let the middle C-edge of f' be xy. Then $vy \notin E(G)$, as otherwise C can be extended by replacing the path (v, w, x, y, z) with (v, y, x, w, d, z). Let f_1 be the face opposite to f that has vw as a C-edge. Since $w(f) < {}^{10}/_{3}$, f_1 is either a minor 3-face with middle C-edge uv or a minor 2-face with C-edges vw and wx. Assume to the contrary that f_1 is a 2-face. Then $vx \notin E(G)$, as otherwise C can be extended by detouring through b. This implies $vz \in E(G)$, as otherwise $\deg_G(v) = 2$. Then $\{w, z\}$ is a 2-separator of G, which is a contradiction.

Hence, f_1 is a 3-face (see Figure 21). Then $ux \notin E(G)$, as otherwise C can be extended by replacing the path (r, u, v, w, x) with (r, b, w, v, u, x). Thus, since $\{w, z\}$ is no 2-separator of G, uy or vx is an edge of G. Assume to the contrary that $uy \notin E(G)$. Then $vx \in E(G)$, and we have $wy \notin E(G)$, as otherwise C can be extended by replacing the path (r, u, v, w, x, y, z) with $(r, b, w, y, x, v, u, a, z)$. Since $\deg_G(y) \geq 3$, this implies $uy \in E(G)$. Assume to the contrary that $vx \notin E(G)$. Then $xz \in E(G)$, as otherwise $\deg_G(x) = 2$, and C can be extended by replacing the path (r, u, v, w, x, y, z) with (r, b, w, v, u, y, x, z), which gives a contradiction. Hence, $uy \in E(G)$ and $vx \in E(G)$. Then C can be extended by replacing the path (u, v, w, x, y, z) with (u, y, x, v, w, d, z).

Case 5e: *f sends weight to an opposite minor 3-face f' with middle C-edge uv or yz, but not to any opposite minor 4- or 5-face* (see Figure 22).

Without loss of generality, let the middle C-edge of f' be yz. Assume first that f sends weight to a second opposite minor 3-face $f_1 \neq f'$.

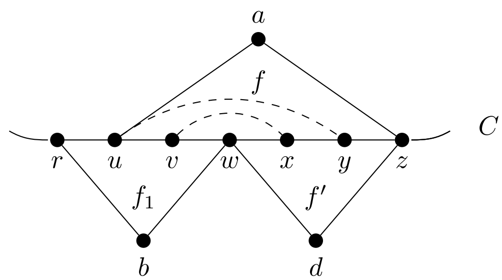

Figure 21: Case 5d

By Case 5d, f_1 has not middle C-edge vw, so that f' must have middle C-edge uv. Then wx is a C-edge of a major face opposite to f that sends weight 1 to f, which contradicts $w(f) < 10/3$.

Hence, in order to satisfy $w(f) < 10/3$, f sends by R2 a total weight of $4/3$ to opposite minor 2-faces. This implies that there is a minor 2-face f_2 opposite to f that satisfies $m_{f,f_2} = 2$. Then f_2 has not C-edges uv and vw, as otherwise wx would once again be a C-edge of a major face, which contradicts $w(f) < 10/3$. Hence, f_2 has C-edges vw and wx (see Figure 22). Then $uw \notin E(G)$, as otherwise C can be extended by replacing the path (u, v, w, x) with (u, w, v, b, x), and $wy \notin E(G)$, as otherwise C can be extended by replacing the path (v, w, x, y) with (v, b, x, w, y). Since $\deg_G(w) \geq 3$, $wz \in E(G)$. Then C can be extended by replacing the path (w, x, y, z, q) with (w, z, y, x, d, q), which is a contradiction.

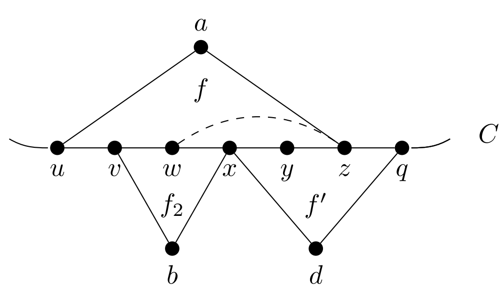

Figure 22: Case 5e

We conclude that f sends no weight to any opposite minor 3-, 4- or 5-face. In order to satisfy $w(f) < 10/3$, f must therefore send a total weight of $6/3$ to opposite minor 2-faces by R2. In particular, there is at least one minor 2-face f' opposite to f that has $m_{f,f'} = 2$. We distinguish the following subcases for f'.

Case 5f: *f' has C-edges uv and vw, or xy and yz* (see Figure 23).

Without loss of generality, let f' have C-edges xy and yz. Assume first that f has a second opposite minor 2-face $f_1 \neq f'$ with $m_{f,f_1} = 2$. Then f_1 has not C-edges uv and vw, as then wx would be a C-edge of a major face sending f weight 1, which implies $w(f) = 5 - 4 \cdot \frac{2}{3} + 1 = 10/3$. Hence, f_1 has C-edges vw and wx (see Figure 23). Then $wy \notin E(G)$, as otherwise C can be extended by replacing the path (w, x, y, z) with (w, y, x, d, z). Hence, $vy \notin E(G)$, as otherwise $\deg_G(w) = 2$. Since $\deg_G(y) \geq 3$, we conclude $uy \in E(G)$ and, by $\deg_G(w) \geq 3$, $uw \in E(G)$. Then C can be extended by replacing the path (u, v, w, x) with (u, w, v, b, x).

Hence, f has no second opposite minor 2-face $f_1 \neq f'$ with $m_{f,f_1} = 2$. Since f sends a total weight of $6/3$ to opposite minor 2-faces by R2, f has an opposite minor 2-face $f_2 \neq f'$ that has C-edge uv but no other C-edge of f. Then vw and wx are C-edges of major face(s), which contradicts $w(f) < 10/3$.

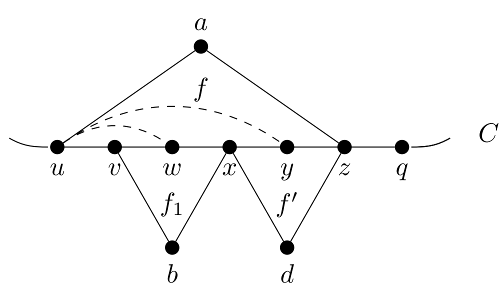

Figure 23: Case 5f

Case 5g: *f' has C-edges vw and wx, or wx and xy* (see Figure 24).

Without loss of generality, let f' have C-edges wx and xy. By Case 5f, f has no second opposite minor 2-face $f_1 \neq f'$ with $m_{f,f_1} = 2$. By $w(f) < 10/3$, f has an opposite minor 2-face f_2 that has exactly one of the C-edges of f as a C-edge. If this edge e is not yz, $e = uv$ and then vw is a C-edge of a major face, which contradicts $w(f) < 10/3$. Hence $e = yz$. Since neither uv nor vw is a C-edge of a major face, as this would again contradict $w(f) < 10/3$, uv and vw are C-edges of a minor j-face f_3 with $j \geq 4$ that does not receive any weight from f. Then f_3 sends weight $1/3$ to f by R5, which gives $w(f) = 10/3$ and thus a contradiction.

Case 6: *f is a minor 6-face* (see Figure 25).

Then f is initially charged with weight 6. If f looses a total net weight of at most $8/3$, then $w(f) \geq 10/3$, so assume that f looses a total net weight of at least $9/3$. We distinguish the following subcases.

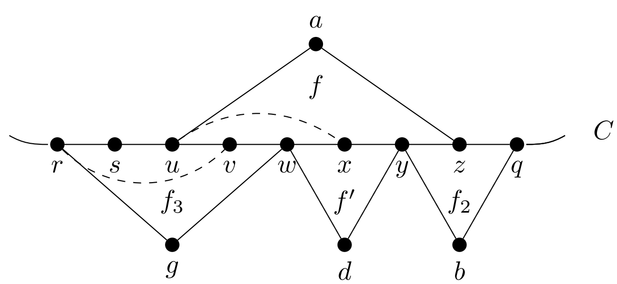

Figure 24: Case 5g

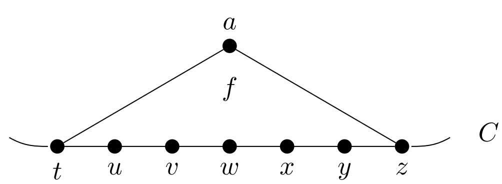

Figure 25: Case 6

Case 6a: *f sends weight to an opposite minor 5-face f'* (see Figure 26). Without loss of generality, let xy and yz be C-edges of f' getting weight from f by R5. Then f sends weight $1/3$ to f', and total weight $8/3$ to opposite minor 2-faces f_3 and f_4 by R1–R5, as otherwise $w(f) \geq 10/3$ (see Figure 26). Let f_1 and f_2 be the two minor 2-faces opposite to f' due to R5.

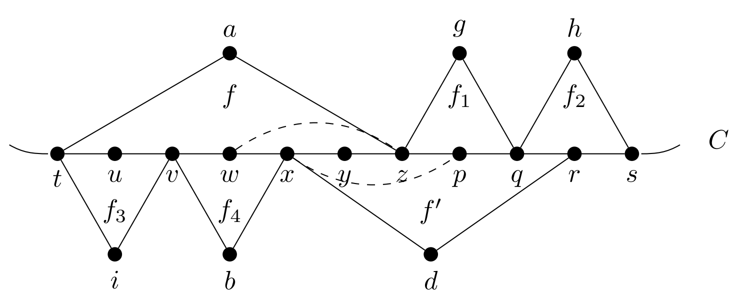

Figure 26: Case 6a

We have $uw \notin E(G)$ and $wy \notin E(G)$, as otherwise C can be extended by detouring through b, and $tw \notin E(G)$, as otherwise $\deg_G(u) = 2$. Since $\deg_G(w) \geq 3$, $wz \in E(G)$. Moreover, $yp \notin E(G)$ and $pr \notin E(G)$, as otherwise C can be extended by detouring through g. Since

$\deg_G(p) \geq 3$, $xp \in E(G)$. Hence, $\deg_G(y) = 2$, which contradicts that G is 3-connected.

Case 6b: *f sends weight to an opposite minor 4-face f'* (see Figure 27).

Without loss of generality, let xy and yz be C-edges of f' by R4. Since $w(f) < 10/3$, f has neither an opposite minor 5-face, nor a second opposite minor 4-face. Assume first that f sends weight to an opposite minor 3-face f_1. Then f sends total weight $5/3$ to f' and f_1, and must therefore send weight $4/3$ to minor 2-face(s), as otherwise $w(f) \geq 10/3$. Hence, f_1 has middle C-edge tu, and f has one opposite minor 2-face f_2 that has C-edges vw and wx (see Figure 27).

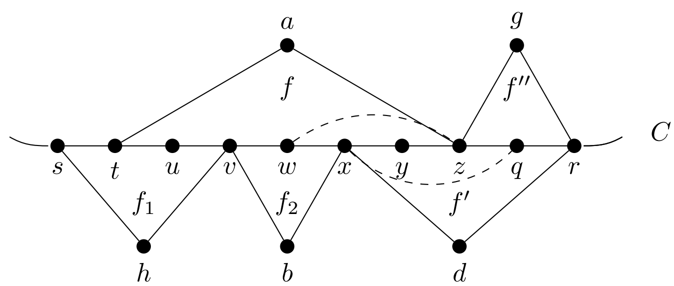

Figure 27: Case 6b

Then uw and wy are not edges of G, as otherwise C can be extended by detouring through b. Moreover, $tw \notin E(G)$, as otherwise C can be extended by replacing the path (s, t, u, v, w) with (s, h, v, u, t, w). Hence, $wz \in E(G)$, as otherwise $\deg_G(w) = 2$. Moreover, $yq \notin E(G)$ and $xq \in E(G)$ for the same reason as in Case 4i, which contradicts $\deg_G(y) \geq 3$.

Case 6c: *f sends weight to an opposite minor 3-face f' with middle C-edge vw or wx* (see Figure 28).

Without loss of generality, let the middle C-edge of f' be wx. In order to have $w(f) < 10/3$, f must by R2–R3 send weight 2 to minor 2-faces. Thus, f has two minor 2-faces f_1 and f_2 such that f_1 has C-edges tu and uv, and f_2 has yz as a C-edge.

Then $uw \notin E(G)$, as otherwise C can be extended by detouring C through b. In addition, $ux \notin E(G)$, as otherwise C can be extended by replacing the path (u, v, w, x, y) with (u, x, w, v, d, y). Then $uy \notin E(G)$, as otherwise the fact that $\{v, y\}$ is not a 2-separator of G would imply $uw \in E(G)$ or $ux \in E(G)$. Since $\deg_G(u) \geq 3$, $uz \in E(G)$. Then we can obtain $\overline{C}$ from C by replacing the path (t, u, v, w, x, y, z, q) with $(t, a, z, u, v, w, x, y, g, q)$.

Case 6d: *f sends weight to an opposite minor 3-face f' with middle C-edge uv or xy* (see Figure 29).

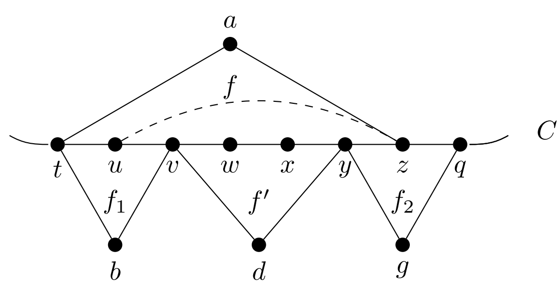

Figure 28: Case 6c

Without loss of generality, let the middle C-edge of f' be xy. As in Case 6c, $w(f) < {}^{10}/_{3}$ implies that f has opposite minor 2-faces f_1 and f_2 such that f_2 has C-edges uv and vw and f_1 has C-edge tu (see Figure 29).

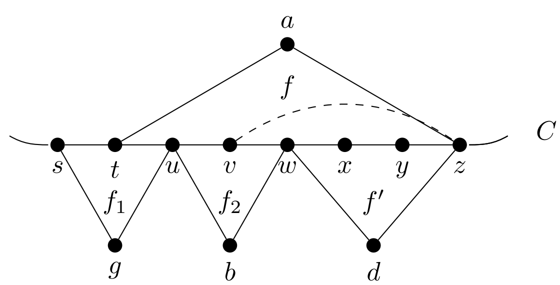

Figure 29: Case 6d

Then tv and vx are not edges of G, as otherwise C can be extended by detouring C through b. In addition, $vy \notin E(G)$, as otherwise C can be extended by replacing the path (v, w, x, y, z) with (v, y, x, w, d, z). Since $\deg_G(v) \geq 3$, $vz \in E(G)$. This implies that $\{w, z\}$ is a 2-separator of G, which contradicts that G is 3-connected.

Case 6e: *f sends weight to an opposite minor 3-face f' with middle C-edge tu or yz, but not to any opposite minor 4- or 5-face* (see Figure 30).

Without loss of generality, let the middle C-edge of f' be yz. Assume first that f has a second opposite minor 3-face f''. By Cases 6c+d, f'' has middle C-edge tu. By $w(f) < {}^{10}/_{3}$, f has an opposite minor 2-face f_2 with C-edges vw and wx (see Figure 30). Then $uw \notin E(G)$ and $wy \notin E(G)$, as otherwise C can be extended by detouring through b. Moreover, $wz \notin E(G)$, as otherwise C can be extended by replacing the path (w, x, y, z, q) with (w, z, y, x, d, q). By symmetry, $tw \notin E(G)$, which contradicts $\deg_G(w) \geq 3$.

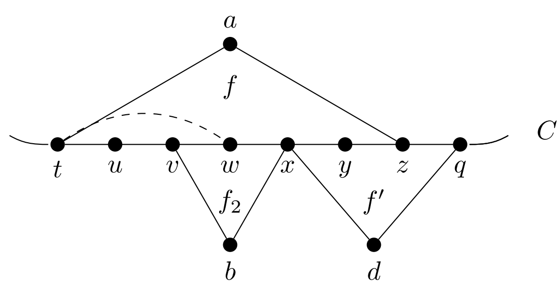

Figure 30: Case 6e

Hence, by R1–R3, f sends total weight 2 to at least two opposite minor 2-faces f_1 and f_2. If $m_{f,f_1} = 1$ or $m_{f,f_2} = 1$, either the edge uv or the edge wx would be a C-edge of a major face, which contradicts $w(f) < {}^{10}/_{3}$. Thus, f_1 has C-edges tu and uv, and f_2 has C-edges vw and wx. From the previous argument, we know that uw, wy and wz are not in G. Since $\deg_G(w) \geq 3$, $tw \in E(G)$. This contradicts $\deg_G(u) \geq 3$.

We conclude that f sends no weight to any opposite minor 3-, 4- or 5-face. In order to satisfy $w(f) < {}^{10}/_{3}$, f must therefore send a total weight of ${}^{10}/_{3}$ to opposite minor 2-faces by R2, as R2 sends only multiples of weight ${}^{2}/_{3}$. If some C-edge e of f is not a C-edge of a minor 2-face, e must be either tu or yz, as otherwise e would be in a major face that sends weight 1 to f and therefore contradicts $w(f) < {}^{10}/_{3}$. Hence, f has three opposite minor 2-faces f_1, f_2 and f_3 such that $m_{f,f_1} = m_{f,f_2} = 2$ and the C-edges of f_1 and f_2 are either uv, vw, wx, xy or one of tu, uv, vw, wx and vw, wx, xy, yz. We distinguish these subcases.

Case 6f: *The C-edges of f_1 and f_2 are tu, uv, vw, wx or vw, wx, xy, yz* (see Figure 31).

Without loss of generality, let f_1 and f_2 have the C-edges vw, wx, xy, yz. By the above argument, f_3 has the C-edges tu and uv (see Figure 31).

Then uw and wy are not in G, as otherwise C can be extended by detouring through b. Moreoever, $wz \notin E(G)$, as otherwise $\deg_G(y) = 2$. By symmetry, $tw \notin E(G)$, which contradicts $\deg_G(w) \geq 3$.

Case 6g: *The C-edges of f_1 and f_2 are uv, vw, wx, xy* (see Figure 32).

Then f_3 has either tu or yz as a C-edge, say without loss of generality the latter.

Then tv and vx are not in G, as otherwise C can be extended by detouring through b. Moreover, $vy \notin E(G)$, as otherwise $\deg_G(x) = 2$. Since $\deg_G(v) \geq 3$, $vz \in E(G)$. Then $xz \notin E(G)$, as otherwise C can be extended by detouring through g. Hence, we obtain the contradiction $\deg_G(x) = 2$.

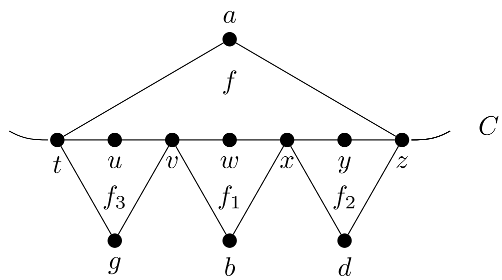

Figure 31: Case 6f

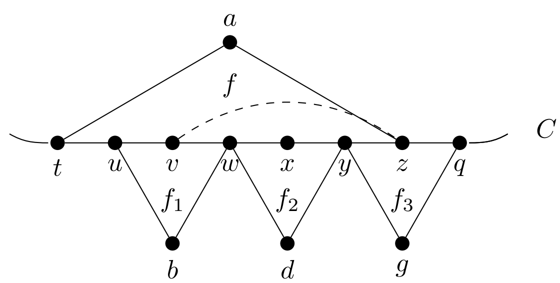

Figure 32: Case 6g

Case 7: *f is a minor 7-face* (see Figure 33).

Then f is initially charged with weight 7. If f looses a total net weight of at most $^{11}/_{3}$, then $w(f) \geq {}^{10}/_{3}$, so assume that f looses a total net weight of at least $^{12}/_{3}$. According to R1–R5, f sends to every opposite face f' at most weight $\frac{2}{3}m_{f,f'}$ (for example, if f' is a minor 3-face, f sends only weight at most $\frac{1}{2}m_{f,f'}$ by R3). Hence, f does not send any weight to a 5-face, as otherwise $w(f) \geq {}^{10}/_{3}$. We distinguish the remaining cases.

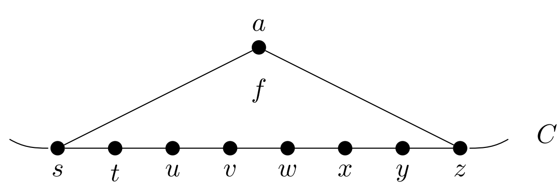

Figure 33: Case 7

Case 7a: *f sends weight to an opposite minor 4-face f'* (see Figure 34).

Without loss of generality, let f' have C-edges xy and yz. Since $w(f) < {}^{10}/_{3}$, all other C-edges of f are C-edges of minor 2-faces f_1,

f_2 and f_3 (see Figure 34).

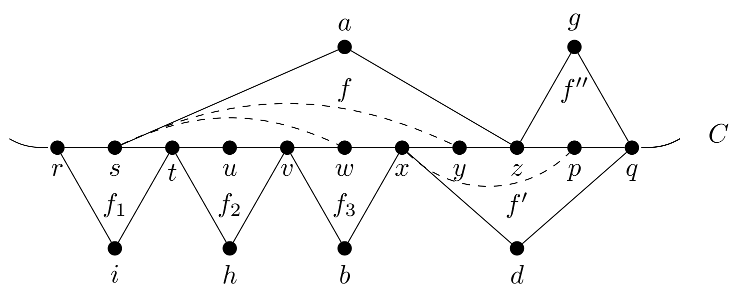

Figure 34: Case 7a

Then $yp \notin E(G)$, as otherwise C can be extended by detouring through g, and hence $xp \in E(G)$, as otherwise $\deg_G(p) = 2$. Also, uw and wy are not in G, as otherwise C can be extended by detouring through b. Hence, y has a neighbor in G that is incident to f and different from $\{w, x, z\}$. We conclude $wz \notin E(G)$. In addition, $tw \notin E(G)$, as otherwise $\deg_G(u) = 2$. Thus, $sw \in E(G)$, which implies $sy \in E(G)$. Then $\overline{C}$ can be obtained from C by replacing the path $(r, s, t, u, v, w, x, y, z)$ with $(r, i, t, u, v, w, x, y, s, a, z)$.

Case 7b: *f sends weight to an opposite minor 3-face f'* (see Figure 35). Since $w(f) < 10/3$, the middle C-edge of f' must be either st or yz; say without loss of generality the latter. For the same reason as in Case 7a, all other C-edges of f are C-edges of minor 2-faces f_1, f_2 and f_3 (see Figure 35). Note that if there is another 3-face f'' with middle C-edge st, then the edges uv, vw and wx are not all C-edges of some 2-face.

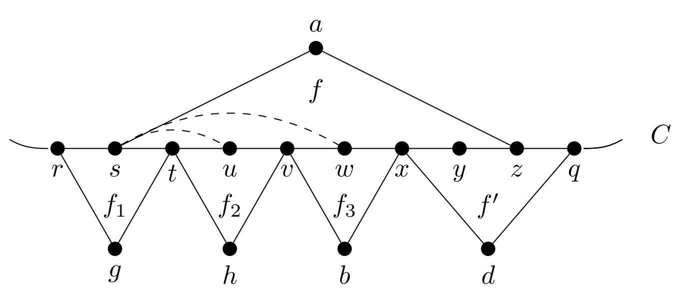

Figure 35: Case 7b

Then $uw \notin E(G)$ and $wy \notin E(G)$, as otherwise C can be extended by detouring through b. Moreover, $wz \notin E(G)$, as otherwise C can be extended by replacing the path (w, x, y, z, q) with (w, z, y, x, d, q).

Also $tw \notin E(G)$, as otherwise $\deg_G(u) = 2$. Since $\deg_G(w) \geq 3$, $sw \in E(G)$. Since $\deg_G(u) \geq 3$, $su \in E(G)$. Then C can be extended by replacing the path (s, t, u, v) with (s, u, t, h, v).

Case 7c: *f sends no weight to 3-, 4- and 5-faces* (see Figure 36).

Then f sends a total weight of at least $6 \cdot \frac{2}{3} = 4$ to opposite minor 2-faces. The C-edges of these 2-faces must be consecutive on C, as otherwise exactly one C-edge of f would be a C-edge of a major face, which contradicts $w(f) < 10/3$. Hence, there are three minor 2-faces f_1, f_2 and f_3, whose C-edges are consecutive on C and satisfy $m_{f,f_1} = m_{f,f_2} = m_{f,f_3} = 2$ (see Figure 36). Assume without loss of generality that f_3 has C-edges xy and yz.

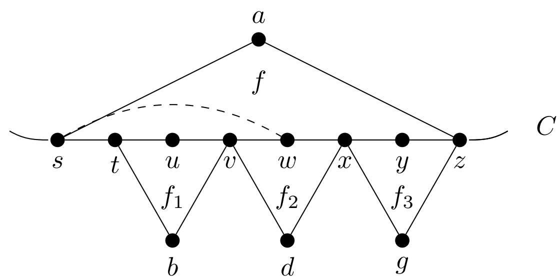

Figure 36: Case 7c

Then uw and wy are not in G, as otherwise C can be extended by detouring through d. Moreover, tw and wz are not in G, as otherwise $\deg_G(u) = 2$ or $\deg_G(y) = 2$. Since $\deg_G(w) \geq 3$, $sw \in E(G)$. Moreover, $su \notin E(G)$, as otherwise C can be extended by detouring through b. Hence, we obtain the contradiction $\deg_G(u) = 2$.

Case 8: *f is a minor 8-face* (see Figure 37).

Then f is initially charged with weight 8. If f looses a total net weight of at most $14/3$, then $w(f) \geq 10/3$, so assume that f looses a total net weight of at least $15/3$. Hence, f does not send any weight to a 4- or 5-face, as otherwise $w(f) \geq 10/3$. We distinguish the remaining cases.

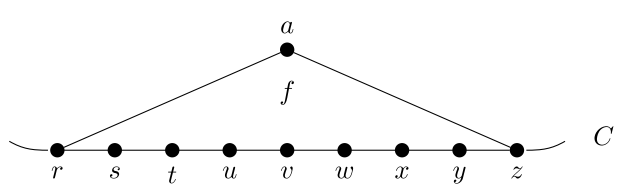

Figure 37: Case 8

Case 8a: *f sends weight to an opposite minor 3-face f'* (see Figure 38).

Then $w(f) < 10/3$ implies that f' has exactly two C-edges that are C-edges of f, and that every other C-edge of f is a C-edge of a minor 2-face. Without loss of generality, let f' have middle C-edge yz, and let f_1, f_2 and f_3 be the minor 2-faces opposite to f (see Figure 38).

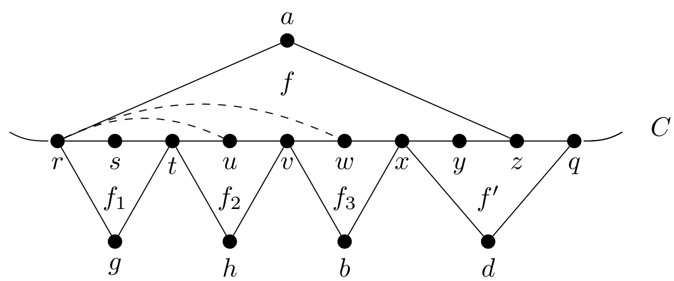

Figure 38: Case 8a

Then su, uw and wy are not edges of G, as otherwise C can be extended by detouring through h or b. Moreover, $wz \notin E(G)$, as otherwise C can be extended by replacing the path (w, x, y, z, q) with (w, z, y, x, d, q). Also $sw \notin E(G)$ and $tw \notin E(G)$, as otherwise $\deg_G(u) = 2$. Since $\deg_G(w) \geq 3$, $rw \in E(G)$. Since $\deg_G(u) \geq 3$, $ru \in E(G)$. This gives the contradiction $\deg_G(s) = 2$.

Case 8b: *f sends no weight to 3-, 4- and 5-faces* (see Figure 39).

Then f sends a total weight of exactly $8 \cdot \frac{2}{3} = 16/3$ to opposite minor 2-faces, as R2 sends only multiples of $\frac{2}{3}$ weight. Assume first that a minor 2-face f_4 opposite to f has C-edges xy and yz (see Figure 39). Then $wy \notin E(G)$, as otherwise C can be extended by detouring through g, and $wz \notin E(G)$, as otherwise $\deg_G(y) = 2$. Then the same arguments as in Case 8a give the contradiction $\deg_G(s) = 2$.

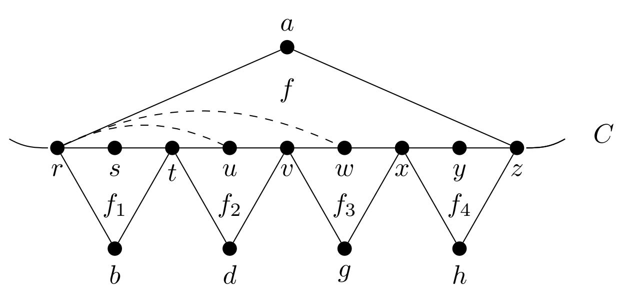

Figure 39: Case 8b

Hence, let yz be the only C-edge of f_4 that is a C-edge of f. Then v

has no neighbor that is incident to f and not in $\{u, w\}$, as otherwise t or x has degree 2 in G. Hence, we obtain the contradiction $\deg_G(v) = 2$.

Case 9: *f is a minor j-face with $j \geq 9$* (see Figure 40)*.*

Then f is initially charged with weight j and looses a total net weight of at most $\frac{2}{3}j$, so that $w(f) \geq \frac{1}{3}j \geq \frac{10}{3}$ if $j \geq 10$. Hence, $j = 9$ and every C-edge of f is a C-edge of a minor 2-face. Since 9 is odd, we may assume without loss of generality that one minor 2-face f_1 has qr but no other C-edge of f as a C-edge (see Figure 40). Then the same arguments as in Cases 8a+b imply that $\deg_G(s) = 2$.

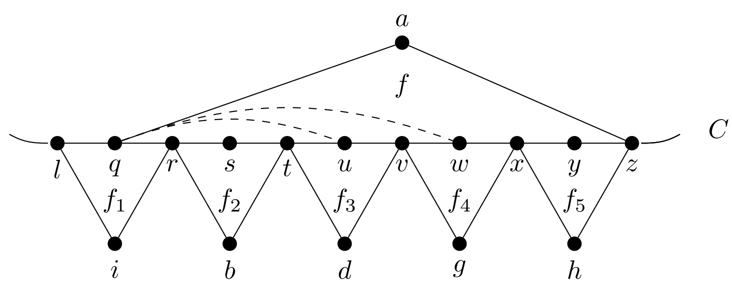

Figure 40: Case 9

This proves $2c = \sum_{f \in F(H)} w(f) \geq {}^{10}\!/_3 \cdot |M^- \cup M^+|$, which completes the proof of Theorem 1. □

4 Remarks

We remark that the bound of Theorem 1 can be improved to $\frac{5}{8}(n+4)$ for every $n \geq 16$: then Lemma 5 in [2] implies the improved bound for the special case that V^- or V^+ is empty, while in the remaining case $|V^-| \geq 1 \leq |V^+|$ Lemma 1 can be immediately strengthened to $|M^- \cup M^+| \geq |V^- \cup V^+| + 4$ using the same proof with a different induction base (see also [3]). This immediately improves the bound $\mathrm{circ}(G) \geq \frac{13}{21}(n+4)$ given in [2] for every $n \geq 16$. We note that $\mathrm{circ}(G) \geq \frac{5}{8}(n+4)$ does not hold for $n \leq 6$, as for these values a cycle of length at least $\frac{5}{8}(n+4) > n$ is impossible.

The proof of Theorem 1 is constructive and gives a quadratic-time algorithm that finds a cycle of length at least $\frac{5}{8}(n+2)$, by applying the result of [6] exactly as shown in [3, Section Algorithm]. We therefore conclude the following theorem.

Theorem 2 *For every essentially 4-connected plane graph G on n vertices, a cycle of length at least $\frac{5}{8}(n+2)$ can be computed in time $O(n^2)$.*

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