Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 22, no. 3, pp. 401–429 (2018)
DOI: 10.7155/jgaa.00475

Aligned Drawings of Planar Graphs

Tamara Mchedlidze [1] *Marcel Radermacher* [1] *Ignaz Rutter* [2]

[1] Department of Computer Science,
Karlsruhe Institute of Technology, Germany
[2] Department of Computer Science and Mathematics,
University of Passau, Germany

Abstract

Let G be a graph that is topologically embedded in the plane and let $\mathcal{A}$ be an arrangement of pseudolines intersecting the drawing of G. An *aligned* drawing of G and $\mathcal{A}$ is a planar polyline drawing Γ of G with an arrangement A of lines so that Γ and A are homeomorphic to G and $\mathcal{A}$. We show that if $\mathcal{A}$ is stretchable and every edge e either entirely lies on a pseudoline or it has at most one intersection with $\mathcal{A}$, then G and $\mathcal{A}$ have a straight-line aligned drawing. In order to prove this result, we strengthen a result of Da Lozzo et al. [5], and prove that a planar graph G and a single pseudoline $\mathcal{L}$ have an aligned drawing with a prescribed convex drawing of the outer face. We also study the less restrictive version of the alignment problem with respect to one line, where only a set of vertices is given and we need to determine whether they can be collinear. We show that the problem is $\mathcal{NP}$-complete but fixed-parameter tractable.

Submitted:	Reviewed:	Revised:	Reviewed:	Revised:
November 2017	April 2018	May 2018	June 2018	July 2018
	Accepted:	Final:	Published:	
	July 2018	July 2018	September 2018	
	Article type:		Communicated by:	
	Regular paper		F. Frati and K.-L. Ma	

Partially supported by grant WA 654/21-1 of the German Research Foundation (DFG). A preliminary version of this paper appeared in the *Proceedings of the 25th International Symposium on Graph Drawing (GD '17)*.

E-mail addresses: mched@iti.uka.de (Tamara Mchedlidze) radermacher@kit.edu (Marcel Radermacher) rutter@fim.uni-passau.de (Ignaz Rutter)

1 Introduction

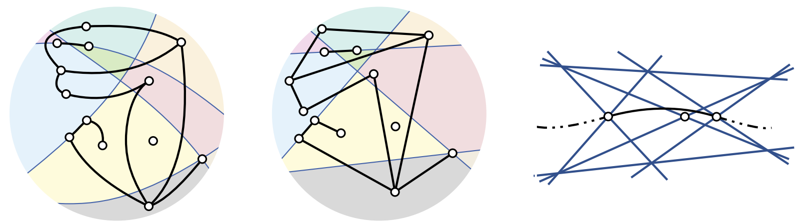
Two fundamental primitives for highlighting structural properties of a graph in a drawing are *alignment* of vertices such that they are collinear, and geometric *separation* of unrelated graph parts, e.g., by a straight line. Both these techniques have been previously considered from a theoretical point of view in the case of planar straight-line drawings.

Da Lozzo et al. [5] study the problem of producing a planar straight-line drawing of a given embedded graph $G = (V, E)$ (i.e., G has a fixed combinatorial embedding and a fixed outer face) such that a given set $S \subseteq V$ of vertices is collinear. It is clear that if such a drawing exists, then the line containing the vertices in S is a simple curve starting and ending at infinity that for each edge e of G either fully contains e or intersects e in at most one point, which may be an endpoint. We call such a curve a *pseudoline with respect to* G. Da Lozzo et al. [5] show that this is a full characterization of the alignment problem, i.e., a planar straight-line drawing where the vertices in S are collinear exists if and only if there exists a pseudoline $\mathcal{L}$ with respect to G that contains the vertices in S. However, the computational complexity of deciding whether such a pseudoline exists is an open problem, which we consider in this paper.

Likewise, for the problem of separation, Biedl et al. [1] considered so-called HH-drawings where, given an embedded graph $G = (V, E)$ and a partition $V = A \cup B$, one seeks a y-monotone planar polyline drawing of G with few bends in which A and B can be separated by a line. Again, it turns out that such a drawing exists if there exists a pseudoline $\mathcal{L}$ with respect to G such that the vertices in A and B are separated by $\mathcal{L}$. As a side-result Cano et al. [2] extend the result of Biedl et al. to planar straight-line drawings with a given star-shaped outer face.

The aforementioned results of Da Lozzo et al. [5] show that given a pseudoline $\mathcal{L}$ with respect to G one can always find a planar straight-line drawing of G such that the vertices on $\mathcal{L}$ are collinear and the vertices contained in the half-planes defined by $\mathcal{L}$ are separated by a line L. In other words, a topological configuration consisting of a planar embedded graph G and a pseudoline with respect to G can always be stretched. In this paper, we initiate the study of this stretchability problem with more than one given pseudoline.

More formally, a pair $(G, \mathcal{A})$ is a *k-aligned graph* if $G = (V, E)$ is a planar embedded graph and $\mathcal{A} = \{\mathcal{L}_1, \dots, \mathcal{L}_k\}$ is an arrangement of (pairwise intersecting) pseudolines with respect to G. In case that every pair of distinct pseudolines intersect at most once, we refer to $\mathcal{A}$ as a *pseudoline arrangement*. If the number k of pseudolines is clear from the context, we drop it from the notation and simply speak of *aligned graphs*. For 1-aligned graphs we write $(G, \mathcal{L})$ instead of $(G, \{\mathcal{L}\})$. Let $A = \{L_1, \dots, L_k\}$ be a line arrangement and Γ be a planar drawing of G. A tuple (Γ, A) is an *aligned drawing of* $(G, \mathcal{A})$ if and only if the arrangement of the union of Γ and A is homeomorphic to the arrangement of the union of G and $\mathcal{A}$. A (pseudo)-line arrangement divides the plane into a set of *cells* $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_\ell$. If A is homeomorphic to $\mathcal{A}$, then there is a bijection ϕ between the cells of $\mathcal{A}$ and the cells of A. If (Γ, A) is an aligned drawing of $(G, \mathcal{A})$,



(a) (b) (c)

Figure 1: (Pseudo-) Lines are depicted as blue curves, edges are black. The color of the cells indicates the bijection ϕ between the cells of $\mathcal{A}$ and A. Aligned drawing (b) of a 2-aligned planar embedded graph (a). (c) A non-stretchable arrangement of 9 pseudolines (blue and black), which can be seen as a stretchable arrangement of 8 pseudolines (blue) and an edge (black solid).

then it has the following properties; refer to Fig. 1(a-b). (i) The arrangement of A is homeomorphic to the arrangement of $\mathcal{A}$ (i.e., $\mathcal{A}$ is *stretchable* to A), (ii) Γ is homeomorphic to the planar embedding of G, (iii) the intersection of each vertex v and each edge e with a cell $\mathcal{C}$ of $\mathcal{A}$ is non-empty if and only if the intersection of v and e with $\phi(\mathcal{C})$ in (Γ, A), respectively, is non-empty, (iv) if an edge uv (directed from u to v) intersects a sequence of cells $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_r$ in this order, then uv intersects in (Γ, A) the cells $\phi(\mathcal{C}_1), \phi(\mathcal{C}_2), \dots, \phi(\mathcal{C}_r)$ in this order, and (v) each line L_i intersects in Γ the same vertices and edges as $\mathcal{L}_i$ in G, and it does so in the same order. We focus on straight-line aligned drawings. For brevity, unless stated otherwise, the term aligned drawing refers to a straight-line drawing throughout this paper.

Note that the stretchability of $\mathcal{A}$ is a necessary condition for the existence of an aligned drawing. Since testing stretchability is $\mathcal{NP}$-hard [13, 15], we assume that a geometric realization A of $\mathcal{A}$ is provided. Line arrangements of size up to 8 are always stretchable [11], and only starting from nine lines non-stretchable arrangements exist; see the Pappus configuration [12] in Fig. 1c. This figure also illustrates an example of an 8-aligned graph with a single edge that does not have an aligned drawing. It is conceivable that in practical applications, e.g., stemming from user interactions, the number of lines to stretch is small, justifying the stretchability assumption.

The aligned drawing convention generalizes the problems studied by Da Lozzo et al. and Biedl et al. who focused on the case of a single line. We study a natural extension of their setting and ask for alignment on general line arrangements.

In addition to the strongly related works mentioned above, there are several other works that are related to the alignment of vertices in drawings. Ravsky and Verbitsky [14] used the fact that 2-trees have a drawing with at least $n/30$ collinear vertices to show that at least $\sqrt{n/30}$ vertices of a 2-tree can be fixed to arbitrary positions. Dujmović [6] shows that every n-vertex planar graph

alignment complexity	k	**drawable**
$(0, \perp, \perp)$	≥ 1	✓ – Planarity
$(0, 0, 0)$	≥ 1	✓ – Theorem 1
$(1, 0, \perp)$	≥ 1	✓ – Theorem 8
$(1, 0, 0)$	2	open – Fig. 17
$(\perp, \perp, 2)$		
$(\perp, 3, \perp)$	≥ 8	✗ – Fig. 1(c)
$(4, \perp, \perp)$		

Table 1: Families of aligned graphs that always have an aligned drawing are marked with ✓. The symbol ✗ indicates that for this particular class, there is an aligned graph that does not have an aligned drawing.

$G = (V, E)$ has a planar straight-line drawing such that $\Omega(\sqrt{n})$ vertices are aligned, and Da Lozzo et al. [5] show that in planar treewidth-3 and planar treewidth-k graphs, one can align $\Theta(n)$ and $\Omega(k^2)$ vertices, respectively. Chaplik et al. [3] study the problem of drawing planar graphs such that all edges can be covered by k lines. They show that it is $\mathcal{NP}$-hard to decide whether such a drawing exists. The computational complexity of deciding whether there exists a drawing where all vertices lie on k lines is an open problem [4]. Drawings of graphs on n lines where a mapping between the vertices and the lines is provided have been studied by Dujmović et al. [7, 8].

Contribution & Outline. After introducing notation in Section 2, we first study the topological setting where we are given a planar graph G and a set S of vertices to align in Section 3. We show that it is $\mathcal{NP}$-complete to decide whether S is alignable. On the positive side, we prove that this problem is fixed-parameter tractable (FPT) with respect to $|S|$. Afterwards, in Section 4, we consider the geometric setting where we seek an aligned drawing of an aligned graph. Based on our proof strategy in Section 4.1, we strengthen the result of Da Lozzo et al. and Biedl et al. in Section 4.2, and show that there exists a 1-aligned drawing of G with a given convex drawing of the outer face. In Section 4.3 we consider k-aligned graphs with a stretchable pseudoline arrangement, where every edge e either entirely lies on a pseudoline or intersects at most one pseudoline, which can either be in the interior or an endpoint of e. We utilize the result of Section 4.2 to prove that every such k-aligned graph has an aligned drawing, for any value of k. In the preliminaries we define the *alignment complexity* of an aligned graph. It is a triple that indicates how many intersections an edge has with the pseudoline arrangement depending on the number of endpoints that lie on a pseudoline. Table 1 summarizes the results of our paper.

2 Preliminaries

Let $\mathcal{A}$ be a pseudoline arrangement with k pseudolines $\mathcal{L}_1, \dots, \mathcal{L}_k$ and $(G, \mathcal{A})$ be an aligned graph with n vertices. The set of cells in $\mathcal{A}$ is denoted by cells($\mathcal{A}$).

A cell is *empty* if it does not contain a vertex of G. Removing from a pseudoline its intersections with other pseudolines gives its *pseudosegments*.

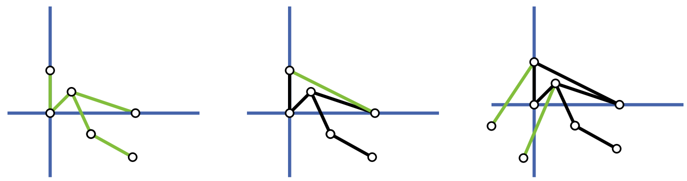
Let $G = (V, E)$ be a planar embedded graph with vertex set V and edge set E. We call $v \in V$ *interior* if v does not lie on the boundary of the outer face of G. An edge $e \in E$ is *interior* if e does not lie entirely on the boundary of the outer face of G. An interior edge is a *chord* if it connects two vertices on the outer face. A point p of an edge e is an *interior* point of e if p is not an endpoint of e. A *triangulation* is a biconnected planar embedded graph whose inner faces are all triangles and whose outer face is bounded by a simple cycle. A *triangulation* of a graph G is a triangulation that contains G as a subgraph. A *k-aligned triangulation* of $(G, \mathcal{A})$ is a k-aligned graph $(G_T, \mathcal{A})$ with G_T being a triangulation of G. A graph G' is a *subdivision* of G if G' is obtained by placing *subdivision vertices* on edges of G. For an abstract graph G and an edge e of G the graph G/e is obtained from G by contracting e and merging the resulting multiple edges and removing self-loops. Routing the edges incident to e close to e yields a planar embedding of G/e in case of a planar embedded graph G. A *k-wheel* is a simple cycle C with k vertices on the outer face and one additional interior vertex that has an edge to each vertex in C. Let Γ be a drawing of G and let C be a cycle in G. We denote with $\Gamma[C]$ the drawing of C in Γ. Let T be a separating triangle in G and let V_{in} and V_{out} be the vertices in the interior and exterior of T, respectively. We refer to the graphs induced by $T \cup V_{\text{in}}$ and $T \cup V_{\text{out}}$ as the *split components of* T and denote them by G_{in} and G_{out}.

A vertex is *$\mathcal{L}_i$-aligned* (or simply *aligned* to $\mathcal{L}_i$) if it lies on the pseudoline $\mathcal{L}_i$. A vertex that is not aligned is *free*. An edge e is *$\mathcal{L}_i$-aligned* (or simply *aligned*) if it completely lies on $\mathcal{L}_i$. Let E_{aligned} be the set of all aligned edges. An *intersection vertex* lies on the intersection of two pseudolines $\mathcal{L}_i$ and $\mathcal{L}_j$. A non-aligned edge is *i-anchored* ($i = 0, 1, 2$) if i of its endpoints are aligned to distinct pseudolines. An $\mathcal{L}$-aligned edge is *i-anchored* ($i = 0, 1, 2$) if i of its endpoints are aligned to distinct pseudolines which are different from $\mathcal{L}$. For example, the single aligned edge in Fig. 2a is 1-anchored. Let E_i be the set of i-anchored edges; note that, the set of edges is the disjoint union $E_0 \mathbin{\dot\cup} E_1 \mathbin{\dot\cup} E_2$. An edge e is (*at most*) *l-crossed* if (at most) l distinct pseudolines intersect e in its interior. A 0-anchored 0-crossed non-aligned edge is also called *free*. A non-empty edge set $A \subset E$ is l-crossed if l is the smallest number such that every edge in A is at most l-crossed.

The *alignment complexity* of an aligned graph describes how "complex" the relationship between the graph G and the pseudoline arrangement $\mathcal{L}_1, \ldots, \mathcal{L}_k$ is. It is formally defined as a triple (l_0, l_1, l_2), where l_i, $i = 0, 1, 2$, indicates that E_i is at most l_i-crossed or has to be empty, if $l_i = \perp$. For example, an aligned graph where every vertex is aligned and every edge has at most l interior intersections has the alignment complexity $(\perp, \perp, l)$. For further examples, see Fig. 2.

Theorem 1 *Every k-aligned graph $(G, \mathcal{A})$ of alignment complexity $(0, 0, 0)$ with a stretchable pseudoline arrangement $\mathcal{A}$ has an aligned drawing.*

Proof: We modify the graph $(G, \mathcal{A})$ as follows; see Fig. 3. We place a vertex on



(a) $(1, 0, \perp)$ (b) $(1, 0, 0)$ (c) $(2, 1, 0)$

Figure 2: Examples for the alignment complexity of an aligned graph.

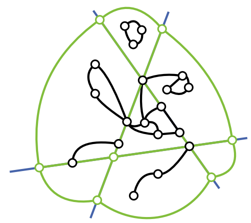


Figure 3: The black edges and vertices and the blue pseudoline arrangement is the input graph $(G, \mathcal{A})$. The green and black graph together depict the modified graph before the triangulation step.

each intersection of two or more pseudolines (if the intersection is not already occupied). In case that k is at least two, every unbounded cell $\mathcal{C}$ of $\mathcal{A}$ has two pseudosegments of infinite length. We place a vertex on each of them at infinity and connect them by an edge routed through the interior of $\mathcal{C}$.

Further, let u and v be two $\mathcal{L}$-aligned vertices, that are consecutive along $\mathcal{L}$. If uv is not already an edge of G, we insert it into G and route it on $\mathcal{L}$. Note that, since $(G, \mathcal{L})$ does not contain edges that cross a pseudoline, the resulting graph is again an aligned graph of alignment complexity $(0, 0, 0)$. The boundary of every cell is covered by aligned edges. Thus, we can triangulate $(G, \mathcal{A})$ without introducing intersections between edges and a pseudoline.

We obtain an aligned drawing of the modified graph as follows. Note that the only interaction between two cells are the aligned vertices and edges on their common boundary, i.e., there are no edges crossing the boundary. Hence, for every pseudosegments of $\mathcal{A}$ we place the aligned vertices on it, arbitrarily (but respecting their order) on the corresponding line segment in A. Since, every cell is covered by aligned edges, we can draw the interior of two cells independently from each other. More formally, the vertex placements of the vertices of the pseudolines prescribes a convex drawing of the outer face of the graph $G_{\mathcal{C}}$, i.e., the graph induced by the vertices in the interior or on the boundary of a cell $\mathcal{C}$. Thus, we obtain a drawing Γ of G by applying the result of Tutte [16] to each graph $G_{\mathcal{C}}$, independently. $\square$

3 Complexity and Fixed-Parameter Tractability

In this section, we deal with the topological setting where we are given a planar embedded graph $G = (V, E)$ and a subset $S \subseteq V$. We ask for a straight-line drawing of G where the vertices in S are collinear. According to Da Lozzo et al. [5], this problem is equivalent to deciding the existence of a pseudoline $\mathcal{L}$ with respect to G passing exactly through the vertices in S. We refer to this problem as *pseudoline existence problem* and the corresponding search problem is referred to as *pseudoline construction problem*. Using techniques similar to Fößmeier and Kaufmann [9], we can show that the pseudoline existence problem is $\mathcal{NP}$-hard.

Let $G^\star + V$ be the graph obtained from the dual graph $G^\star = (V^\star, E^\star)$ of $G = (V, E)$ by placing every vertex $v \in V$ in its dual face $v^\star$ and connecting it to every vertex on the boundary of the face $v^\star$.

Lemma 1 *Let $G = (V, E)$ be a 3-connected 3-regular planar graph. There exists a pseudoline through V with respect to the graph $G^\star + V$ if and only if G is Hamiltonian.*

Proof: Recall that the dual of a 3-connected 3-regular graph is a triangulation with a single combinatorial embedding.

Assume that there exists a pseudoline $\mathcal{L}$ through V with respect to $G^\star + V$. Then the order of appearance of the vertices of $G^\star + V$ on $\mathcal{L}$ defines a sequence of adjacent faces in $G^\star$, i.e., vertices of the primal graph G that are connected via primal edges. This yields a Hamiltonian cycle in G.

Let C be a Hamiltonian cycle of G and consider a simultaneous embedding of G and $G^\star + V$ on the plane, where each pair of a primal and its dual edge intersects exactly once. Thus, the cycle C crosses each dual edge e at most once and passes through exactly the vertices V. There is a vertex v on the cycle C such that v lies in the unbounded face of $G^\star + V$. Thus, the cycle C can be interpreted as a pseudoline $\mathcal{L}(V)$ in $G^\star + V$ through all vertices in V by splitting it in the unbounded face of $G^\star + V$. □

Since computing a Hamiltonian cycle in 3-connected 3-regular planar graphs is $\mathcal{NP}$-complete [10], we get that the pseudoline construction problem is $\mathcal{NP}$-hard. On the other hand, we can guess a sequence of vertices, edges and faces of G, and then test in polynomial time whether this corresponds to a pseudoline $\mathcal{L}$ with respect to G that traverses exactly the vertices in S. Thus, the pseudoline construction problem is in $\mathcal{NP}$. This proves the following theorem.

Theorem 2 *The pseudoline existence problem is $\mathcal{NP}$-complete.*

In the following, we show that the pseudoline construction problem is fixed-parameter tractable with respect to $|S|$. To this end, we construct a graph $G^{\mathrm{tr}} = (V^{\mathrm{tr}}, E^{\mathrm{tr}})$ and a set $S^{\mathrm{tr}} \subseteq V^{\mathrm{tr}}$ with $|S^{\mathrm{tr}}| \leq |S| + 1$ such that G^{tr} contains a simple cycle traversing all vertices in S^{tr} if and only if there exists a pseudoline $\mathcal{L}$ that passes exactly through the vertices in S such that $(G, \mathcal{L})$ is an aligned graph.

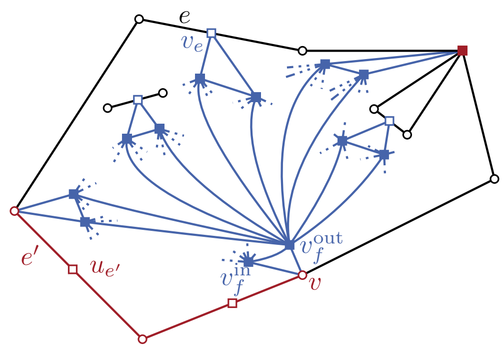


Figure 4: The black and red edges depict a single face of the input graph G. Red and blue edges build the transformed graph G^{tr}. Red round vertices are vertices in S, red squared vertices illustrate the set S^{tr}, the filled red square is a vertex in S and S^{tr}. Blue dashed edges sketch the clique edges between clique vertices (filled blue).

We observe that if the vertices S of a positive instance are not independent, they can only induce a *linear forest*, i.e., a set of paths, as otherwise, there is no pseudoline through all the vertices in S with respect to G. We call the edges on the induced paths *aligned edges*. An edge that is not incident to a vertex in S is called *crossable*, in the sense that only crossable edges can be crossed by $\mathcal{L}$, otherwise $\mathcal{L}$ is not a pseudoline with respect to G. Let $S_{\text{ep}} \subseteq S$ be the subset of vertices that are endpoints of the paths induced by S (an isolated vertex is a path of length 0). We construct G^{tr} in several steps; refer to Fig. 4.

Step 1 Let G' be the graph obtained from G by subdividing each aligned edge e with a new vertex u_e and let S^{tr} be the set consisting of all isolated vertices in S and the new subdivision vertices. Additionally, we add to G' one new vertex o that we embed in the outer face of G and also add to S^{tr}. Observe that by construction $|S^{\text{tr}}| \leq |S| + 1$. Finally, subdivide each crossable edge e by a new vertex v_e. We call these vertices *traversal nodes* and denote their set by $T = S_{\text{ep}} \cup \{v_e \mid e \text{ is crossable}\} \cup \{o\}$. Intuitively, a curve will correspond to a path that uses the vertices in S_{ep} to hop onto paths of aligned edges and the subdivision vertices of crossable edges to traverse from one face to another. Moreover, the vertex $o \in S^{\text{tr}}$ plays a similar role, forcing the curve to visit the outer face.

Step 2 For each face f of G' we perform the following construction. Let $T(f)$ denote the traversal nodes that are incident to f. For each vertex $v \in T(f)$ we create two new vertices v_f^{in} and v_f^{out}, add the edges vv_f^{in} and vv_f^{out} to G', and draw them in the interior of f. Finally, we create a clique $C(f)$ on the vertex set $\{v_f^{\text{in}}, v_f^{\text{out}} \mid v \in T(f)\}$, and embed its edges in the interior of f.

Step 3 To obtain G^{tr} remove all edges of G' that correspond to edges of G

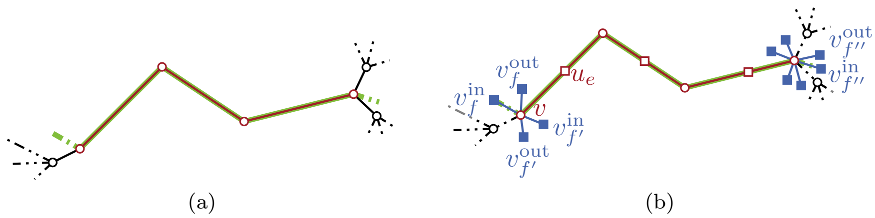


Figure 5: (a) A pseudoline (thick green) traversing a path of aligned edges (thin red). (b) A path (thick green) in $G^{\rm tr}$ visiting consecutive vertices in $S^{\rm tr}$ (red squared).

except those that stem from subdividing an aligned edge of G.

Lemma 2 *There exists a pseudoline $\mathcal{L}$ traversing exactly the vertices in S such that $(G, \mathcal{L})$ is an aligned graph if and only if there exists a simple cycle in $G^{\rm tr}$ that traverses all vertices in $S^{\rm tr}$.*

Proof: Suppose C is a cycle in $G^{\rm tr}$ that visits all vertices in $S^{\rm tr}$. Without loss of generality, we assume that there is no face f such that C contains a subpath from $v_f^{\rm in}$ via v to $v_f^{\rm out}$ (or its reverse) for some vertex $v \in T(f) \setminus S_{\rm ep}$, as otherwise we simply shortcut this path by the edge $v_f^{\rm in} v_f^{\rm out} \in C(f)$.

Consider a path P of aligned edges in G that contains at least one edge; refer to Fig. 5. By definition, C visits all the subdivision vertices $u_e \in S^{\rm tr}$ of the edges of P, and thus it enters P on an endpoint of P, traverses P and leaves P at the other endpoint. All isolated vertices of S are contained in $S^{\rm tr}$, and therefore C indeed traverses all vertices in S (and thus also all aligned edges). As described above, $G^{\rm tr}$ is indeed a topological graph, and thus C corresponds to a closed curve ρ that traverses exactly the vertices in S and the aligned edges.

We now show that ρ can be transformed to a pseudoline with respect to G. Let e be a non-aligned edge of G that has a common point with ρ in its interior;

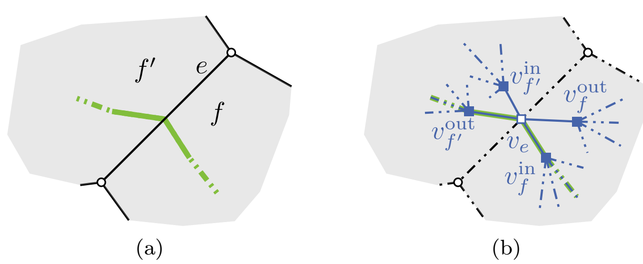


Figure 6: (a) A pseudoline (thick green) passing through a non-aligned edge. (b) A path (thick green) in $G^{\rm tr}$ traversing a subdivision vertex v_e (blue non-filled square). Black (dashed) segments are edges of G.

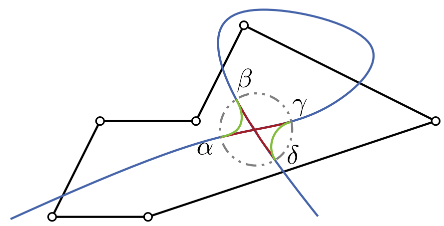


Figure 7: Resolving an intersection by exchanging the intersecting segments (red) with non-intersecting segments (green).

see Fig. 6. Thus, C contains the subdivision vertex v_e. In particular, this implies that e is crossable. Moreover, from our assumption on C, it follows that C enters v_e via v_f^{in} or v_f^{out} and leaves it via $v_{f'}^{\text{in}}$ or $v_{f'}^{\text{out}}$, where f and f' are the faces incident to e, and it is $f \neq f'$ as we could shortcut C otherwise. Therefore, ρ indeed intersects e and uses it to traverse to a different face of G. Moreover, since e has only a single subdivision vertex in G^{tr} and C is simple, it follows that e is intersected only once. Thus ρ is a curve that intersects all vertices in S, traverses all aligned edges, and crosses each edge of G (including the endpoints) at most once. Moreover, ρ traverses the outer face since C contains o.

The only reasons why ρ is not necessarily a pseudoline with respect to G are that it is a closed curve and it may cross itself. However, we can break ρ in the outer face and route both ends to infinity, and remove such self-intersections locally as follows; see Fig. 7. Consider a circle D around an intersection I that neither contains a second self-intersection nor a vertex, nor an edge of G. Let $\alpha, \beta, \gamma, \delta$ be the intersections of D with $\mathcal{L}$. We replace the pseudosegment $\alpha\gamma$ with a pseudosegment $\alpha\beta$, and $\beta\delta$ with a pseudosegment $\gamma\delta$. We route the pseudosegments $\alpha\beta$ and $\gamma\delta$ through the interior of D such that they do not intersect. Thus, we obtain a pseudoline $\mathcal{L}$ with respect to G that contains exactly the vertices in S.

For the converse assume that $\mathcal{L}$ is a pseudoline that traverses exactly the vertices in S such that $(G, \mathcal{L})$ is an aligned graph. The pseudoline $\mathcal{L}$ can be split into three parts $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ such that $\mathcal{L}_1$ and $\mathcal{L}_3$ have infinite length and do not intersect with G, and $\mathcal{L}_2$ has its endpoints in the outer face of G. We transform $\mathcal{L}$ into a closed curve $\mathcal{L}'$ by removing $\mathcal{L}_1, \mathcal{L}_3$ and adding a new piece connecting the endpoints of $\mathcal{L}_2$ without intersecting $\mathcal{L}_2$ or G. Additionally, we choose an arbitrary direction for $\mathcal{L}'$ in order to determine an order of the crossed edges and vertices.

We show that G^{tr} contains a simple cycle traversing the vertices in S^{tr}. By definition $\mathcal{L}'$ consists of two different types of pieces, see Fig. 5. The first type traverses a path of aligned edges between two vertices in S_{ep}. The other type traverses a face of G by entering and exiting it either via an edge or from a vertex in S_{ep}; see Fig. 8. We show how to map these pieces to paths in G^{tr}; the cycle C is obtained by concatenating all these paths.

Each piece of the first type indeed corresponds directly to a path in G^{tr};

see Fig. 5. Consider now a piece π of the second type traversing a face f; refer to Fig. 8. The piece π enters f either from a vertex in S_{ep} or by crossing a crossable edge e. In either case, $T(f)$ contains a corresponding traversal node u. Likewise, $T(f)$ contains a traversal node v for the edge or vertex that $\mathcal{L}'$ intersects next. We map π to the path $uu_f^{\mathrm{in}}v_f^{\mathrm{out}}v$ in G^{tr}. By construction, paths corresponding to consecutive pieces of $\mathcal{L}'$ share a traversal node, and therefore concatenating all paths yields a cycle C in G^{tr}. Moreover, C is simple, since $\mathcal{L}'$ intersects each edge and each vertex at most once. Note that C contains at least one edge of the outer face (as $\mathcal{L}'$ traverses the outer face), and we modify C so that it also traverses the special vertex o.

It remains to show that C contains all vertices in S^{tr}. There are three types of vertices in S^{tr}; the subdivision vertices of aligned edges, the isolated vertices in S, and the special vertex o. The latter is in C by the last step of the construction. The isolated vertices in S are traversed by $\mathcal{L}'$ and contained in S_{ep}, and they are therefore visited also by C. Finally, the subdivision vertices of aligned edges are traversed by the paths corresponding to the first type of pieces, since $\mathcal{L}'$ traverses all aligned edges. □

Theorem 3 (Wahlström [17]) *Given an n-vertex graph $G = (V, E)$ and a subset $S \subseteq V$, it can be tested in $O(2^{|S|}\mathrm{poly}(n))$ time whether a simple cycle through the vertices in S exists. If affirmative the cycle can be reported within the same asymptotic time.*

Theorem 4 *The pseudoline construction problem is solvable in $O(2^{|S|}\mathrm{poly}(n))$ time, where n is the number of vertices.*

Proof: Let $G = (V, E)$ with $S \subseteq V$ be an instance of the pseudoline construction problem. By Lemma 2 the pseudoline construction problem is equivalent to determining whether G^{tr} contains a simple cycle visiting all vertices in S^{tr}. Since the size of G^{tr} is $O(n^2)$ and it can be constructed in $O(n^2)$ time, and $|S^{\mathrm{tr}}| \leq |S| + 1$, Theorem 3 can be used to solve the latter problem in the desired running time. □

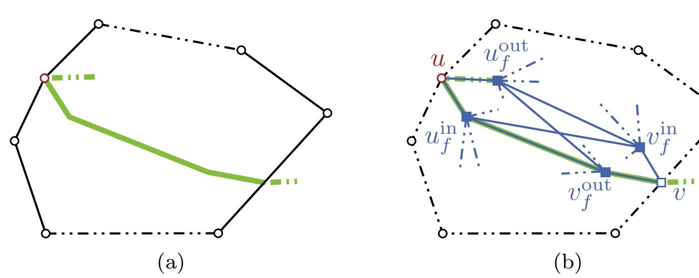


Figure 8: (a) A pseudoline piece π (thick green) passing through a face f. (b) Path (thick green) in G^{tr} corresponding to π.

We note that indeed the construction of G^{tr} only allows leaving a path of aligned edges at an endpoint in S_{ep}. Therefore, a single vertex in S^{tr} for each path of aligned edges would be sufficient to ensure that C traverses the whole path. Thus, by removing for each path all but one vertex from S^{tr} we obtain an algorithm that is FPT with respect to the number of paths induced by S.

Theorem 5 *The pseudoline construction problem is solvable in $O(2^P \text{poly}(n))$ time, where n is the number of vertices and P is the number of paths induced by the vertex set S to be aligned.*

4 Drawing Aligned Graphs

We show that every aligned graph where each edge either entirely lies on a pseudoline or is intersected by at most one pseudoline, i.e., alignment complexity $(1, 0, \perp)$, has an aligned drawing. For 1-aligned graphs we show the stronger statement that every 1-aligned graph has an aligned drawing with a given aligned convex drawing of the outer face. We first present our proof strategy and then deal with 1- and k-aligned graphs.

4.1 Proof Strategy

Our general strategy for proving the existence of aligned drawings of an aligned graph $(G, \mathcal{A})$ is as follows. First, we show that we can triangulate $(G, \mathcal{A})$ by adding vertices and edges without invalidating its properties. We can thus assume that our aligned graph $(G, \mathcal{A})$ is an aligned triangulation. Second, we show that unless G has a specific structure (e.g., a k-wheel or a triangle), it contains an aligned or a free edge. Third, we exploit the existence of such an edge to reduce the instance. Depending on whether the edge is contained in a separating triangle or not, we either decompose along that triangle or contract the edge. In both cases the problem reduces to smaller instances that are almost independent. In order to combine solutions, it is, however, crucial to use the same arrangement of lines A for both of them.

In the following, we introduce the necessary tools used for all three steps on k-aligned graphs of alignment complexity $(1, 0, \perp)$. Recall, that for this class (i) every non-aligned edge is at most 1-crossed, (ii) every 1-anchored edge is 0-crossed, and (iii) there is no edge with its endpoints on two pseudolines.

Lemmas 3 – 5 show that every aligned graph of alignment complexity $(1, 0, \perp)$ has an aligned triangulation with the same alignment complexity. If G contains a separating triangle, Lemma 6 shows that $(G, \mathcal{A})$ admits an aligned drawing if both split components have an aligned drawing. Finally, with Lemma 7 we obtain a drawing of $(G, \mathcal{A})$ from a drawing of the aligned graph $(G/e, \mathcal{A})$ where one particular edge e is contracted.

Lemma 3 *Let $(G, \mathcal{A})$ be a k-aligned n-vertex graph of alignment complexity $(1, 0, \perp)$. Then there exists a biconnected k-aligned graph $(G', \mathcal{A})$ that contains*

G as a subgraph. The set $E(G') \setminus E(G)$ has alignment complexity $(1, 0, \perp)$ and does not contain aligned edges. The size of $E(G') \setminus E(G)$ is in $O(nk + k^3)$.

Proof: Our procedure works in two steps. First, we connect disconnected components. Second, we assure that the graph is biconnected by inserting edges around a cut-vertex. Initially, we place a vertex in every cell that does not contain a vertex in its interior.

Consider a cell $\mathcal{C}$ of $\mathcal{A}$ that contains two vertices u and v that belong to distinct connected components G_u and G_v. We refer to two vertices u, v that lie in the interior or on the boundary of $\mathcal{C}$ as *$\mathcal{C}$-visible* if there is a curve in the interior of $\mathcal{C}$ that connects u to v and that does not intersect G except at its endpoints. In the following, we exhaustively connect $\mathcal{C}$-visible pairs of vertices of distinct connected components of G. If u and v are $\mathcal{C}$-visible, we simply connect them by an edge e. In case that both vertices are aligned, we have to subdivide the edge e with a vertex to avoid introducing 2-anchored edges to the graph. Assume that u, v are not $\mathcal{C}$-visible. Consider any curve ρ in the interior of $\mathcal{C}$ that connects u and v. Then ρ intersects a set of edges of G either in their interior or in a vertex. Thus, there are two edges e_1 and e_2 consecutive along ρ, that belong two distinct connected components. Since e_1 and e_2 are at most 1-crossed, there is an endpoint of e_1 and an endpoint of e_2 that are $\mathcal{C}$-visible and thus can be connected by an edge. Overall it is sufficient to add a linear number of edges to join distinct connected components that have vertices in a common cell.

By construction, every cell contains at least one free vertex. Thus, in order to connect the graph we consider two cells $\mathcal{C}_1, \mathcal{C}_2$ with a common boundary. Assume that there is a vertex u on the common boundary. In this case, the previous step ensures that there is a path from u to every vertex that lies in the interior or on the boundary of $\mathcal{C}_1$ or $\mathcal{C}_2$. Hence, consider the case where no vertex lies on the common boundary of the two cells. Moreover, the common boundary does also not contain an edge, since this edge would be 2-anchored or l-crossed, $l \geq 2$. Similar to the previous step, we can connect two arbitrary vertices of $\mathcal{C}_1$ and $\mathcal{C}_2$ with a curve ρ that intersects the common boundary. If this curve does not intersect an edge we can simply connect the two vertices with an edge. Otherwise, at least in one cell $\mathcal{C}' \in \{\mathcal{C}_1, \mathcal{C}_2\}$ the curve intersects at least one edge. Therefore, there is an edge e' that comes immediately before the intersection of ρ with the boundary of $\mathcal{C}'$. Since every edge is at most 1-crossed, there are two vertices in $\mathcal{C}_1$ and $\mathcal{C}_2$ that can be connected by an edge. Due to the previous step, we can assume that the vertices in the interior of each cell are connected by a path. Thus, we add at most one edge for each pair of adjacent cells. Since there are $O(k^2)$ cells we add $O(k^2)$ vertices and edges to G, i.e., the size of G is $O(n + k^2)$.

We now assume that G is connected but not biconnected and has $n' \in O(n + k^2)$ vertices. Consider a single cut vertex v; refer to Fig. 9. We consider the common arrangement $\mathcal{F}$ of $\mathcal{A}$ and G, i.e., a face can be restricted by pseudosegments of $\mathcal{A}$ and edges of G. Let $\mathcal{F}_v$ be the set of faces in $\mathcal{F}$ with v on their boundary. We place a vertex v_f in every face f of $\mathcal{F}_v$. Let f and f' be two

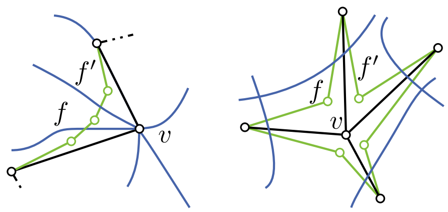


Figure 9: Green edges and vertices are added around a cut-vertex v to connect the connected components (black) incident to v. (a) v is an intersection vertex. (b) v is a free vertex.

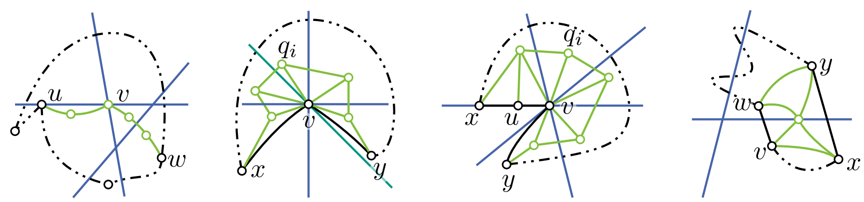


Figure 10: Black lines indicate a face f of G. Light green edges or vertices are newly added into f. Blue lines denote the pseudoline arrangement. (a) Isolation of an intersection. (b-c) Isolation of an aligned vertex or edge. (d) Isolation of a pseudosegment.

distinct faces of $\mathcal{F}_v$ with a common edge ϵ on their boundary. If ϵ is an edge uv of G, we insert the edges uv_f and $uv_{f'}$. Since uv is at most 1-crossed, the new edges are as well at most 1-crossed. If ϵ corresponds to a pseudosegment, we insert the edge $v_f v_{f'}$ such that it crosses ϵ. Since v_f and $v_{f'}$ are free vertices, the edge is by construction 1-crossed.

This procedure adds $O(k + \deg v)$ vertices and edges around v, since at most k pseudolines intersect in a single point. The degree of vertices adjacent to v is increased by at most 2. Thus, the size of G increases to $O(n'k)$. Thus, we have that the size of G is $O(nk + k^3)$. □

Lemma 4 *Let $(G, \mathcal{A})$ be a biconnected k-aligned n-vertex graph of alignment complexity $(1, 0, \bot)$. There exists a k-aligned triangulation $(G_T = (V_T, E_T), \mathcal{A})$ of f whose size is $O(nk + k^3)$. The set $E(G_T) \setminus E(G)$ has alignment complexity $(1, 0, \bot)$ and does not contain aligned edges.*

Proof: We call a face *non-triangular* if its boundary contains more than three vertices. An aligned vertex v or an aligned edge e is *isolated* if all faces with v or e on their boundaries are triangles. A pseudosegment s is *isolated* if s does not intersect the interior of a simple cycle. Our proof distinguishes four cases. Each case is applied exhaustively in this order.

1. If the interior of f contains the intersection of two or more pseudolines, we split the face so that there is a vertex that lies on the intersection.
2. If the boundary of a face has an aligned vertex or an aligned edge, we isolate the vertex or the edge from f.
3. If the interior of a face f intersects a pseudoline $\mathcal{L}$, then it subdivides $\mathcal{L}$ into a set of pseudosegments. We isolate each of the pseudosegments independently.
4. Finally, if none of the previous cases apply, i.e., neither the boundary nor the interior of f contains parts of a pseudoline, the face f can be triangulated with a set of additional free edges.

Let $\mathcal{A}_f$ be the arrangement of $\mathcal{A}$ restricted to the interior of f.

1. Let f be a non-triangular face whose interior contains an intersection of two or more pseudolines; see Fig. 10a. We place a vertex on every intersection in the interior of f. We obtain a biconnected graph G_1 with the application of Lemma 3. Since there are $O(k^2)$ intersections, the size of G_1 is $O\left((n+k^2)k+k^3\right) = O(nk+k^3)$.
2. Let f_1 be a non-triangular face of G_1 with an aligned vertex or an aligned edge uv on its boundary. Further, the interior of f_1 does not contain the intersection of a set of pseudolines; see Fig. 10b and 10c. In case of an aligned vertex we simply assume $u = v$. Since G is biconnected, there exist two edges xu, vy on the boundary of f_1. Let $\mathcal{C}_1, \dots, \mathcal{C}_l \in \text{cells}(\mathcal{A}_{f_1})$ be cells with u or v on their boundary, such that $\mathcal{C}_i$ is adjacent to $\mathcal{C}_{i+1}, i < l$. Since f_1 does not contain 2-anchored edges, at most one of the vertices u and v can be an intersection vertex. Thus, l is at most $2k$. We construct an aligned graph $(G_2, \mathcal{A})$ from $(G_1, \mathcal{A})$ as follows. We place a vertex q_i in the interior of each cell $\mathcal{C}_i, i \le l$. Let $q_0 = x$ and $q_{l+1} = y$. We insert edges $e_i = q_i q_{i+1}, i = 0, \dots, l$ in the interior of f_1 so that the interior of e_i crosses the common boundary of $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ exactly once and it crosses no other boundary. Thus, if the edge e_i is either incident to x or to y, it at most 1-anchored and 0-crossed. Otherwise, it is 0-anchored and 1-crossed. The added path splits f into two faces f', f'' with a unique face f' containing u and v on its boundary. If $w \in \{u, v\}$ is on the boundary of cell $\mathcal{C}_i$, we insert an edge wq_i. Each edge wq_i is 1-anchored and 0-crossed. Let $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ be two cells incident to w. Then, the vertices w, q_i, q_{i+1} form a triangle. If $u \neq v$, there is a unique cell $\mathcal{C}_i$ incident to u and v. Hence, the vertices u, v, q_i form a triangle. Moreover, for $1 \le i \le l$, every edge uq_i and vq_i is incident to two triangles. Therefore, f' is triangulated. By construction, we do not insert aligned vertices and edges, thus the number of aligned edges and aligned vertices of f'' is one less compared to f_1. Hence, we can inductively proceed on f''.

   Assume the aligned vertex v is an intersection vertex. Thus, isolating v uses $O(k)$ additional vertices and edges. Therefore, all intersection vertices can be isolated with $O(k^3)$ vertices and edges.

   Now consider an aligned vertex v that is not an intersection vertex. In this case v is incident to at most two cells. We can isolate all such vertices with $O(n)$ vertices and edges. The same bound holds for aligned edges. Finally, we obtain an aligned graph $(G_2, \mathcal{A})$ of size $O(nk + k^3)$.

3. Let f_2 be a non-triangular face of G_2 whose interior intersects a pseudoline $\mathcal{L}$ and has no aligned edge and no aligned vertex on its boundary. Further, the interior of f_2 does not contain the intersection of two or more pseudolines. Then the face f_2 subdivides $\mathcal{L}$ into a set of pseudosegments; see Fig. 10d. We iteratively isolate such a pseudosegment $\mathcal{S}$. Since f_2 does not contain the intersection of two or more pseudolines in its interior, there are two distinct cells $\mathcal{C}_1 \in \text{cells}(\mathcal{A}_f)$ and $\mathcal{C}_2 \in \text{cells}(\mathcal{A}_f)$ with $\mathcal{S}$ on their boundary. Since f_1 neither contains an aligned vertex nor an aligned edge and G is biconnected, there are exactly two edges $e_1 = vw$ and $e_2 = xy$ with the endpoints of $\mathcal{S}$ in the interior of these edges and v, x and w, y on the boundaries of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Since f_2 does not have an l-crossed edge, $l \geq 2$, and every 1-crossed edge is 0-anchored, the vertices v, w, x, y are free. We construct a graph G' by placing a vertex u on s and inserting edges uv, uw, ux uy, vx and wy. We route each edge so that the interior of an edge does not intersect the boundary of a cell $\mathcal{C}_i, i = 1, 2$. Thus, the edges vx and wy are free and the others are 1-anchored and 0-crossed.
   Every edge in G_2 is at most 1-crossed, thus the number of pseudosegments is linear in the size of G_2. Therefore, we add a number of vertices and edges that is linear in the size of G_2.
   Thus, we obtain an aligned graph $(G_3, \mathcal{A})$ of size $O(nk + k^3)$.
4. If none of the cases above applies to a non-triangular face f_4 of G_3, then neither the interior nor the boundary of the face intersects a pseudoline $\mathcal{L}_i$. Thus, we can triangulate f_4 with a number of free edges linear in the size of f_4. Thus, in total we obtain an aligned triangulation $(G_T, \mathcal{A})$ of $(G, \mathcal{A})$ of size $O(nk + k^3)$.

□

Observe that the correctness of the previous triangulation procedure only relies on the fact that every non-triangular face contains at most 1-crossed edges. While Lemma 4 is sufficient for our purposes, for the sake of generality, we show how to isolate l-crossed edges. This allows us to triangulate biconnected aligned graphs without increasing the alignment complexity.

Theorem 6 *Every biconnected k-aligned n-vertex graph $(G, \mathcal{A})$ of alignment complexity (l_0, l_1, l_2) has an aligned triangulation $(G_T, \mathcal{A})$. The alignment complexity of $E(G_T) \backslash E(G)$ is $(\max\{l_0, 1\}, l_1, l_2)$ and the size of this set is $O(nk + k^3)$.*

Proof: For $l \geq 1$, we iteratively isolate l-crossed edges uv from a non-triangular face f as sketched in Fig. 11. Let $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_l \in \text{cells}(\mathcal{A})$ be the cells in f that occur in this order along uv. If one of these vertices is free, say v, we place a new vertex x in the interior of $\mathcal{C}_{l-1}$. We insert the two edges ux, xv and route both edges close to uv. This isolates the edge uv from f. Notice that the edge xv is 0-anchored and 1-crossed and the edge ux $(l-1)$-crossed. In case

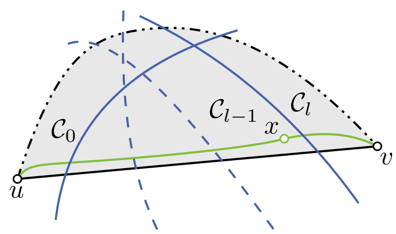


Figure 11: An l-crossed edge uv in a (grey) face f and a pseudoline arrangement (blue). The green edges isolate the edge uv.

that $l_0 \geq 1$, the alignment complexity of the new aligned graph is (l_0, l_1, l_2). Otherwise, the alignment complexity is $(1, l_1, l_2)$. If u and v are aligned, we place x on the boundary of $\mathcal{C}_{l-1}$ and $\mathcal{C}_l$ and route the edges ux and vx as before. The alignment complexity is not affected by this operation. The face uvx is triangular and therefore the edge uv is processed as above at most twice.

This procedure introduces a new $(l-1)$-crossed edge. Repeating the process $l-2$ times generates a new face f' from f where edge uv is substituted by a path of at most 1-crossed edges. To isolate all l-crossed edges in $(G, \mathcal{A})$, we add $O(kn)$ vertices and edges.

By isolating all l-crossed edges in this way, we obtain an aligned graph where every non-triangular face is bounded by at most 1-crossed edges. The proof of Lemma 4 handles all non-triangular faces independently. For the correctness of the triangulation it is sufficient to ensure that every non-triangular face does neither contain 2-anchored edges nor l-crossed edges. Thus, we can apply the methods used in the proof of Lemma 4 to triangulate $(G, \mathcal{A})$ with $O(nk + k^3)$ additional vertices and edges. □

We now return to the treatment of aligned graphs with alignment complexity $(1, 0, \perp)$. To simplify the proofs, we augment the input graph with an additional cycle in the outer face that contains all intersections of $\mathcal{A}$ in its interior, and we add subdivision vertices on the intersections of $\mathcal{L}_i$-aligned edges with pseudolines $\mathcal{L}_j$, $i \neq j$. A k-aligned graph is *proper* if (i) every aligned edge is 0-crossed, (ii) for $k \geq 2$, every edge on the outer face is 1-crossed, (iii) the boundary of the outer face intersects every pseudoline exactly twice, and (iv) the outer face does not contain any intersection of $\mathcal{A}$.

An aligned graph $(G_{\text{rs}}, \mathcal{A})$ is a *rigid subdivision* of an aligned graph $(G, \mathcal{A})$ if and only if G_{rs} is a subdivision of G and every subdivision vertex is an intersection vertex with respect to $\mathcal{A}$. We show that we can extend every k-aligned graph $(G, \mathcal{A})$ to a proper k-aligned triangulation.

Lemma 5 *For every $k \geq 2$ and every k-aligned n-vertex graph $(G, \mathcal{A})$ of alignment complexity $(1, 0, \perp)$, let $(G_{\text{rs}}, \mathcal{A})$ be a rigid subdivision of $(G, \mathcal{A})$. Then there exists a proper k-aligned triangulation $(G', \mathcal{A})$ of alignment complexity $(1, 0, \perp)$ such that G_{rs} is a subgraph of G'. The size of G' is in $O(nk^2 + k^4)$.*

The set $E(G') \setminus E(G_{\text{rs}})$ *has alignment complexity* $(1, 0, \perp)$ *and does not contain aligned edges.*

Proof: We construct a rigid subdivision $(G_{\text{rs}}, \mathcal{A})$ from $(G, \mathcal{A})$ by placing subdivision vertices on the intersections of $\mathcal{L}_i$-aligned edges with pseudolines $\mathcal{L}_j, i \neq j$. The number n_{rs} of vertices of G_{rs} is in $O(n + k^2)$.

We obtain a proper biconnected k-aligned graph $(G_b, \mathcal{A})$ by embedding a simple cycle C in the outer face of G_{rs} and applying Lemma 3. In order to construct C, we place a vertex v_c in each unbounded cell c of $\mathcal{A}$ and connect two vertices v_c and $v_{c'}$ if the boundaries of the cells c and c' intersect. The size n_b of G_b is $O(n_{\text{rs}}k + k^3) = O(nk + k^3)$. We obtain a proper k-aligned triangulation $(G', \mathcal{A})$ of G_b with the application of Lemma 4. The size n' of G' is in $O(n_b k + k^3) = O((nk + k^3)k + k^3) = O(nk^2 + k^4)$. □

The following two lemmas show that we can reduce the size of the aligned graph and obtain a drawing by merging two drawings or by geometrically uncontracting an edge.

Lemma 6 *Let* $(G, \mathcal{A})$ *be a* k*-aligned triangulation. Let* T *be a separating triangle splitting* G *into subgraphs* $G_{\text{in}}, G_{\text{out}}$ *so that* $G_{\text{in}} \cap G_{\text{out}} = T$ *and* G_{out} *contains the outer face of* G*. Then, (i)* $(G_{\text{out}}, \mathcal{A})$ *and* $(G_{\text{in}}, \mathcal{A})$ *are* k*-aligned triangulations, and (ii)* $(G, \mathcal{A})$ *has an aligned drawing if and only if there exists a common line arrangement* A *such that* $(G_{\text{out}}, \mathcal{A})$ *has an aligned drawing* (Γ_{out}, A) *and* $(G_{\text{in}}, \mathcal{A})$ *has an aligned drawing* (Γ_{in}, A) *with the outer face drawn as* $\Gamma_{\text{out}}[T]$*.*

Proof: It is easy to verify that $(G_{\text{out}}, \mathcal{A})$ and $(G_{\text{in}}, \mathcal{A})$ are aligned triangulations. An aligned drawing (Γ, A) of $(G, \mathcal{A})$ immediately implies the existence of an aligned drawing (Γ_{out}, A) of $(G_{\text{out}}, \mathcal{A})$ and (Γ_{in}, A) of $(G_{\text{in}}, \mathcal{A})$.

Let (Γ_{out}, A) be an aligned drawing of $(G_{\text{out}}, \mathcal{A})$. Since (Γ_{out}, A) is an aligned drawing, $(\Gamma_{\text{out}}[T], A)$ is an aligned drawing of $(T, \mathcal{A})$. Let (Γ_{in}, A) be an aligned drawing of $(G_{\text{in}}, \mathcal{A})$ with the outer face drawn as $\Gamma_{\text{out}}[T]$. Let Γ be the drawings obtained by merging the drawing Γ_{out} and Γ_{in}. Since (Γ_{out}, A) and (Γ_{in}, A) are aligned drawings on the same line arrangement A, (Γ, A) is an aligned drawing of $(G, \mathcal{A})$. □

Lemma 7 *Let* $(G, \mathcal{A})$ *be a proper* k*-aligned triangulation of alignment complexity* $(1, 0, \perp)$ *and let* e *be an interior* 0*-anchored aligned edge or an interior free edge of* G *that does not belong to a separating triangle and is not a chord. Then* $(G/e, \mathcal{A})$ *is a proper* k*-aligned triangulation of alignment complexity* $(1, 0, \perp)$*. Further,* $(G, \mathcal{A})$ *has an aligned drawing if* $(G/e, \mathcal{A})$ *has an aligned drawing.*

Proof: We first prove that $(G/e, \mathcal{A})$ is a proper k-aligned triangulation. Consider a topological drawing of the aligned graph $(G, \mathcal{A})$. Let c be the vertex in G/e obtained from contracting the edge $e = uv$. We place c at the position of u. Thus, all the edges incident to u keep their topological properties. We route the edges incident to v close to the edge uv within the cell from which they arrive

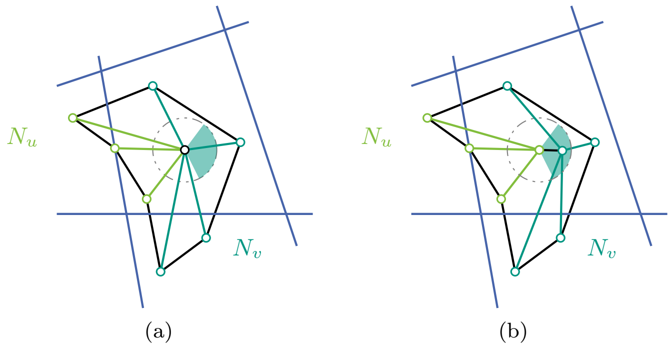


Figure 12: Unpacking an edge in a drawing Γ' of G/e (a) to obtain a drawing Γ of G (b).

to v in $(G, \mathcal{A})$. Since e is not an edge of a separating triangle, G/e is simple and triangulated.

Consider a free edge e. Observe that the triangular faces incident to e do not contain an intersection of two pseudolines in their interior, since $(G, \mathcal{A})$ does not contain l-crossed edges, for $l \geq 2$. Therefore, $(G/e, \mathcal{A})$ is an aligned triangulation. Since e is not a chord, $(G/e, \mathcal{A})$ is proper. Further, u and v lie in the interior of the same cell, thus, the edges incident to c have the same alignment complexity as in $(G, \mathcal{A})$.

If e is aligned, it is also 0-crossed, since $(G, \mathcal{A})$ is proper. Since e is also 0-anchored, the triangles incident to e do not contain an intersection of two pseudolines and therefore $(G/e, \mathcal{A})$ is a proper aligned triangulation. The routing of the edges incident to c, as described above, ensures that the alignment complexity is $(1, 0, \perp)$.

Let (Γ', A) be an aligned drawing of $(G/e, \mathcal{A})$. We now prove that $(G, \mathcal{A})$ has an aligned drawing. Let Γ'' denote the drawing obtained from Γ' by removing c together with its incident edges and let f denote the face of Γ'' where c used to lie. Since G/e is triangulated and e is an interior edge and not a chord, f is star-shaped and c lies inside the kernel of f; see Fig. 12. We construct a drawing Γ of G as follows. If one of vertices u and v lies on the outer face, we assume, without loss of generality, that vertex to be u. First, we place u at the position of c and insert all edges incident to u. This results in a drawing of the face f' in which we have to place v. Since u is placed in the kernel of f, f' is star-shaped. If e is a free edge, the vertex v has to be placed in the same cell as u. We then place v inside f' sufficiently close to c so that it lies inside the kernel of f' and in the same cell as u. All edges incident to v are at most 1-crossed, thus, (Γ, A) is an aligned drawing of $(G, \mathcal{A})$.

Likewise, if e is an $\mathcal{L}$-aligned edge, then v has to be placed on the line $L \in A$ corresponding to $\mathcal{L}$. In this case, also c and therefore u lie on L. Since e is an interior edge, there exist two triangles uv, vx, xu and uv, vy, yu sharing the edge

uv. Since, e is not part of a separating triangle, x and y are on different sides of L. Therefore the face f' contains a segment of the line L of positive length that is within the kernel of f'. Thus, we can place v close to u on the line L such that the resulting drawing is an aligned drawing of $(G, \mathcal{A})$. □

Note that contracting a 1-anchored aligned edge can result in a graph $(G/e, \mathcal{A})$ with an alignment complexity that does not coincide with the alignment complexity of $(G, \mathcal{A})$. Further, for general alignment complexities there is an aligned graph $(G, \mathcal{A})$ and an 1-anchored aligned edge e such that $(G/e, \mathcal{A})$ is not an aligned graph.

4.2 One Pseudoline

We show that every 1-aligned graph $(G, \mathcal{R})$ has an aligned drawing (Γ, R), where $\mathcal{R}$ is a single pseudoline and R is the corresponding straight line. Using the techniques from the previous section, we can assume that $(G, \mathcal{R})$ is a proper 1-aligned triangulation. We show that unless G is very small, it contains an edge with a certain property. This allows for an inductive proof to construct an aligned drawing of $(G, \mathcal{R})$.

Lemma 8 *Let $(G, \mathcal{R})$ be a proper 1-aligned triangulation without chords and with k vertices on the outer face. If G is neither a triangle nor a k-wheel whose center is aligned, then $(G, \mathcal{R})$ contains an interior aligned or an interior free edge.*

Proof: We first prove two useful claims.

Claim 1. Consider the order in which $\mathcal{R}$ intersects the vertices and edges of G. If vertices u and v are consecutive on $\mathcal{R}$, then the edge uv is in G and aligned.

Observe that the edge uv can be inserted into G without creating crossings. Since G is a triangulation, it therefore already contains uv, and further, since every non-aligned edge has at most one of its endpoints on $\mathcal{R}$, it follows that indeed uv is aligned. This proves the claim. ◁

Claim 2. If $(G, \mathcal{R})$ is an aligned triangulation without aligned edges and x is an interior free vertex of G, then x is incident to a free edge.

Assume for a contradiction that all neighbors of x lie either on $\mathcal{R}$ or on the other side of $\mathcal{R}$. First, we slightly modify $\mathcal{R}$ to a curve $\mathcal{R}'$ that does not contain any vertices. Assume v is an aligned vertex; see Fig. 13. Since there are no aligned edges, $\mathcal{R}$ enters v from a face f incident to v and leaves it to a different face f' incident to v. We then reroute $\mathcal{R}$ from f to f' locally around v. If v is incident to x, we choose the rerouting such that it crosses the edge vx.

Notice that if an edge e intersects $\mathcal{R}$ in its endpoints, then $\mathcal{R}'$ either does not intersect it or intersects it in an interior point. Moreover, e cannot intersect $\mathcal{R}'$ twice as in such a case $\mathcal{R}$ would pass through both its endpoints. Now, since G is a triangulation and the outer face of G is proper, $\mathcal{R}'$ corresponds to a simple cycle in the dual $G^\star$ of G, and hence corresponds to a cut C of G. Let H denote the connected component of $G - C$ that contains x and note that all

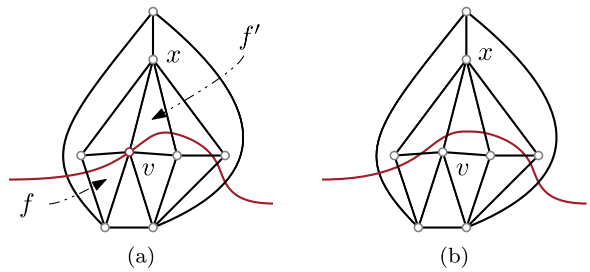


Figure 13: Transformation from a red vertex (a) to a gray vertex (b).

edges of H are free. By the assumption and the construction of $\mathcal{R}'$, x is the only vertex in H. Thus, $\mathcal{R}'$ intersects only the faces incident to x, which are interior. This contradicts the assumption that $\mathcal{R}'$ passes through the outer face of G and finishes the proof of the claim. ◁

We now prove the lemma. Assume that G is neither a triangle nor a k-wheel whose center is aligned. If G is a k-wheel whose center is free, we find a free edge by Claim 2. Otherwise, G contains at least two interior vertices. If one of these vertices is free, we find a free edge by Claim 2. Otherwise, all interior vertices are aligned. Since G does not contain any chord, there is a pair of aligned vertices consecutive along $\mathcal{R}$. Thus by Claim 1 the instance $(G, \mathcal{R})$ has an aligned edge. □

Theorem 7 *Let $(G, \mathcal{R})$ be a proper aligned graph and let (Γ_O, R) be a convex aligned drawing of the aligned outer face $(O, \mathcal{R})$ of G. There exists an aligned drawing (Γ, R) of $(G, \mathcal{R})$ with the same line R and the outer face drawn as Γ_O.*

Proof: Given an arbitrary proper aligned graph $(G, \mathcal{R})$, we first complete it to a biconnected graph and then triangulate it by applying Lemma 3 and Lemma 4, respectively.

We prove the claim by induction on the size of G. If G is just a triangle, then clearly (Γ_O, R) is the desired drawing. If G is the k-wheel whose center is aligned, placing the vertex on the line in the interior of Γ_O yields an aligned drawing of G. This finishes the base case.

If G contains a chord e, then e splits $(G, \mathcal{R})$ into two graphs G_1, G_2 with $G_1 \cap G_2 = e$. It is easy to verify that $(G_i, \mathcal{R})$ is an aligned graph. Let (Γ^i_O, R) be a drawing of the face of $\Gamma_O \cup e$ whose interior contains G_i. By the inductive hypothesis, there exists an aligned drawing of (Γ_i, R) with the outer face drawn as (Γ^i_O, R). We obtain a drawing Γ by merging the drawings Γ_1 and Γ_2. The fact that both (Γ_1, R) and (Γ_2, R) are aligned drawings with a common line R and compatible outer faces implies that (Γ, R) is an aligned drawing of $(G, \mathcal{R})$.

If G contains a separating triangle T, let G_{in} and G_{out} be the respective split components with $G_{\text{in}} \cap G_{\text{out}} = T$. By Lemma 6, the graphs $(G_{\text{in}}, \mathcal{R})$ and $(G_{\text{out}}, \mathcal{R})$ are aligned graphs. By the induction hypothesis there exists an aligned drawing (Γ_{out}, R) of the aligned graphs $(G_{\text{out}}, \mathcal{R})$ with the outer face

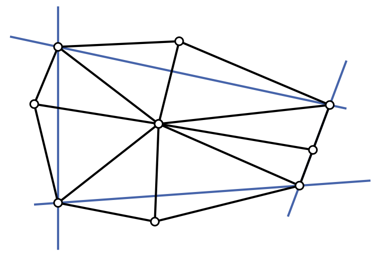


Figure 14: All possible variations of vertices and edges in Lemma 9.

drawn as (Γ_O, R). Let $\Gamma[T]$ be the drawing of T in Γ_{out}. Further, $(G_{\text{in}}, \mathcal{R})$ has by induction hypothesis an aligned drawing with the outer face drawn as $\Gamma[T]$. Thus, by Lemma 6 we obtain an aligned drawing of $(G, \mathcal{R})$ with the outer face drawn as Γ_O.

If G is neither a triangle nor a k-wheel, by Lemma 8, it contains an interior aligned or an interior free edge e. Since e is not a chord and does not belong to a separating triangle, by Lemma 7, $(G/e, \mathcal{R})$ is an aligned graph and by the induction hypothesis it has an aligned drawing (Γ', R) with the outer face drawn as Γ_O. It thus follows by Lemma 7 again that $(G, \mathcal{R})$ has an aligned drawing with the outer face drawn as Γ_O. □

4.3 Alignment Complexity $(1, 0, \perp)$

We now consider k-aligned graphs $(G, \mathcal{A})$ of alignment complexity $(1, 0, \perp)$, i.e., every edge with two free endpoints intersects at most one pseudoline, every 1-anchored edge has no interior intersection with a pseudoline, and 2-anchored edges are entirely forbidden. In this section, we prove that every such k-aligned graph has an aligned drawing. As before we can assume that $(G, \mathcal{A})$ is a proper aligned triangulation. We show that if the structure of the graph is not sufficiently simple, it contains an edge with a special property. Further, we prove that every graph with a sufficiently simple structure indeed has an aligned drawing. Together this again enables an inductive proof that $(G, \mathcal{A})$ has an aligned drawing. Fig. 14 illustrates the statement of the following lemma.

Lemma 9 *For $k \geq 2$ let $(G, \mathcal{A})$ be a proper k-aligned triangulation of alignment complexity $(1, 0, \perp)$ that neither contains a free edge, nor a 0-anchored aligned edge, nor a separating triangle. Then (i) every intersection contains a vertex, (ii) every cell of the pseudoline arrangement contains exactly one free vertex, (iii) every pseudosegment is either covered by two aligned edges or it intersects a single edge.*

Proof:

The statement follows from the following sequence of claims. We refer to an aligned vertex that is not an intersection vertex as a *flexible aligned* vertex.

Claim 1. Every intersection contains a vertex.

Assume that there is an intersection I that does not contain a vertex. Since $(G, \mathcal{A})$ is proper, every aligned edge of G is 0-crossed. Thus, no edge of G contains I in its interior. Moreover, since $(G, \mathcal{A})$ is a proper triangulation, the outer face of G does not contain intersections of $\mathcal{A}$. Hence, there is a triangular face f of G that is not the outer face and that contains I. Thus, f either has a 2-anchored edge, a 1-anchored l_1-crossed edge, $l_1 \geq 1$, or an l_0-crossed edge, $l_0 \geq 2$, on its boundary. This contradicts that $(G, \mathcal{A})$ has alignment complexity $(1, 0, \perp)$.

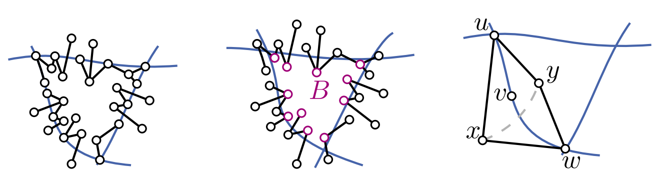
Claim 2. Every cell contains at least one free vertex.

Let $\mathcal{C}$ be a cell of $\mathcal{A}$. Assume that the boundary of $\mathcal{C}$ is neither covered by 1-aligned edges nor crossed by an edge. Since $(G, \mathcal{A})$ is proper, there is a face f of G that entirely contains $\mathcal{C}$ in its interior. Further, G is triangulated and therefore, f is a triangle. But every triangle that contains a cell $\mathcal{C}$ in its interior either has a 2-anchored edge, a 1-anchored l_1-crossed edge, $l_1 \geq 1$, or an l_0-crossed edge, $l_0 \geq 2$, on its boundary. The alignment complexity of $(G, \mathcal{A})$ excludes these types of edges, thus, there is either a 1-crossed edge with an interior intersection with the boundary of $\mathcal{C}$, or $\mathcal{C}$ is covered by 1-anchored aligned edges.

If there is an edge e with an interior intersection with the boundary of $\mathcal{C}$, one endpoint of e lies in the interior of $\mathcal{C}$. Thus, in the following we can assume that no such edges exist. Therefore, the boundary of $\mathcal{C}$ is covered by 1-anchored aligned edges. There are two possibilities to triangulate the interior of the cell, either by edges routed through the interior of $\mathcal{C}$ with endpoints on the boundary of $\mathcal{C}$ or with interior vertices. The former is not possible, since such a non-aligned edge would either be 2-anchored or have both of its endpoints on the same pseudoline. Since $(G, \mathcal{A})$ is an aligned graph of alignment complexity $(1, 0, \perp)$, it does not contain such edges. Thus, every proper aligned triangulation of the graph induced by edges on the boundary of $\mathcal{C}$ contains a vertex in the interior of $\mathcal{C}$. ◁

Claim 3. Every cell contains at most one free vertex.

The following proof is similar to Claim 2 in the proof of Lemma 8. Let $\mathcal{C}$ be a cell and assume for the sake of a contradiction that $\mathcal{C}$ contains more than one vertex in its interior; see Fig. 15a. These vertices are connected by a set of edges to adjacent cells. If $\mathcal{C}$ contains a vertex v or an edge e on its boundary, we



(a) (b) (c)

Figure 15: Illustrations for the proof Lemma 9.

reroute the corresponding pseudolines close to v and e, respectively, such that v and e are now outside of $\mathcal{C}$; refer to Fig. 15b. Let $\mathcal{C}'$ be the resulting cell, it represents a cut in the graph with two components A and B, where $\mathcal{C}'$ contains B in its interior. It is not difficult to see that the modified pseudolines are still pseudolines with respect to G. Since $(G, \mathcal{A})$ neither contains 2-anchored edges, nor 1-anchored l_1-crossed edges, $l_1 \geq 1$, nor l_0-crossed edges, $l_0 \geq 2$, every edge of $(G, \mathcal{A}')$ intersects the boundary of $\mathcal{C}'$ at most once. Further, G is a triangulation and therefore, B is connected and since it contains at least two vertices it also contains at least one free edge, contradicting our initial assumption. ◁

Claim 4. Every flexible aligned vertex is incident to two 1-anchored aligned edges.

Let v be a flexible aligned vertex that lies on a pseudosegment $\mathcal{S}$ of $\mathcal{A}$; refer to Fig. 15c. Since $k \geq 2$, $\mathcal{S}$ is either incident to one or two intersection vertices. Let u be an intersection vertex incident to $\mathcal{S}$ and let $\mathcal{S}$ be on the boundary of the cells $\mathcal{C}_1, \mathcal{C}_2$. First, we will show that u is adjacent to a vertex x in the interior of $\mathcal{C}_1$ and a vertex y in the interior of $\mathcal{C}_2$, respectively. Depending on whether $\mathcal{S}$ is incident to one or two intersection vertices, the edge ux helps to find either a separating triangle or a 4-cycle that each contains v in its interior.

We initially show that the graph contains the edge ux. Since G is triangulated there is a fan of triangles around u. Further, all edges in $(G, \mathcal{A})$ are at most 1-crossed, hence we find a vertex x' in the interior of $\mathcal{C}_1$. Due to Claim 3 and Claim 4 the vertex in the interior of $\mathcal{C}_1$ is unique. Thus, we have that x' is equal to x and therefore G contains the edge ux. Correspondingly, we find a vertex y in the interior of $\mathcal{C}_2$ adjacent to u.

Consider the case where $\mathcal{S}$ contains only a single intersection vertex, i.e, $\mathcal{S}$ intersects the outer face of G. Since $(G, \mathcal{A})$ is proper (edges on the outer face are 1-crossed), G contains the edge xy. Thus, we find a triangle with the vertices x, y and u that contains v in its interior. This contradicts the assumption that G does not have a separating triangle. Therefore, if $\mathcal{S}$ is incident to a single intersection, there is no flexible aligned vertex that lies in the interior of $\mathcal{S}$.

Now consider the case where $\mathcal{S}$ is incident to two intersection vertices u and w. As shown before, the vertices u, w are each adjacent to the free vertices x and y. Therefore, vertices u, w, x, y build a 4-cycle containing v in its interior. Since G does not contain a separating triangle, it cannot contain the edge xy. Moreover, v is the only vertex in the interior of $\mathcal{S}$, as otherwise, we would find a free aligned edge. Finally, since $(G, \mathcal{A})$ is an aligned triangulation, the vertex v is connected to all four vertices and thus v is incident to two 1-anchored aligned edges. ◁

Claim 1 proves that $(G, \mathcal{A})$ has Property (i). Claim 2 and Claim 3 together prove that Property (ii) is satisfied. Since $(G, \mathcal{A})$ is an aligned triangulation, Property (iii) immediately follows from Property (ii) and Claim 4. □

Lemma 10 *Let $(G, \mathcal{A})$ be a proper k-aligned triangulation of alignment complexity $(1, 0, \perp)$ that does neither contain a free edge, nor a 0-anchored aligned* *edge, nor a separating triangle. Let A be a line arrangement homeomorphic to the pseudoline arrangement $\mathcal{A}$. Then $(G, \mathcal{A})$ has an aligned drawing (Γ, A).*

Proof: We obtain a drawing (Γ, A) by placing every free vertex in its cell, every aligned vertex on its pseudosegment and every intersection vertex on its intersection. According to Lemma 9 every cell and every intersection contains exactly one vertex and each pseudosegment is either crossed by an edge or it is covered by two aligned edges. Observe that the union of two adjacent cells of the arrangement A is convex. Thus, this drawing of G has an homeomorphic embedding to $(G, \mathcal{A})$ and every edge intersects in (Γ, A) the line $L \in A$ corresponding to the pseudoline $\mathcal{L} \in \mathcal{A}$ in $(G, \mathcal{A})$ □

We prove the following theorem along the same lines as Theorem 7.

Theorem 8 *Every k-aligned graph $(G, \mathcal{A})$ of alignment complexity $(1, 0, \perp)$ with a stretchable pseudoline arrangement $\mathcal{A}$ has an aligned drawing.*

Proof: Let $(G, \mathcal{A})$ be an arbitrary aligned graph, such that $\mathcal{A}$ is a stretchable pseudoline arrangement, let us denote by A the corresponding line arrangement. By Lemma 5, we obtain a proper k-aligned triangulation $(G_T, \mathcal{A})$ that contains a rigid subdivision of G as a subgraph. Assume that $(G_T, \mathcal{A})$ has an aligned drawing (Γ_T, A). Let (Γ', A) be the drawing obtained from (Γ_T, A) by removing all subdivision vertices v and merging the two edges incident to v at the common endpoint. Recall that a subdivision vertex in a rigid subdivision of $(G, \mathcal{A})$ lies on an intersection in $\mathcal{A}$. Hence the drawing (Γ', A) is a straight-line aligned drawing and contains an aligned drawing (Γ, A) of $(G, \mathcal{A})$.

We now show that $(G_T, \mathcal{A})$ indeed has an aligned drawing. We prove this by induction on the size of the instance $(G_T, \mathcal{A})$. If $(G_T, \mathcal{A})$ neither contains a free edge, nor a 0-anchored aligned edge, nor a separating triangle, then, by Lemma 10 there is an aligned drawing (Γ_T, A).

If G contains a separating triangle T, let G_{in} and G_{out} be the respective split components with $G_{\text{in}} \cap G_{\text{out}} = T$. Since the alignment complexity of $(G, \mathcal{A})$ is $(1, 0, \perp)$, triangle T is intersected by at most one pseudoline $\mathcal{L}$. It follows that $(G_{\text{out}}, \mathcal{A})$ is a k-aligned triangulation and that $(G_{\text{in}}, \mathcal{L})$ is a 1-aligned triangulation. By the induction hypothesis there exists an aligned drawing (Γ_{out}, A) of $(G_{\text{out}}, \mathcal{A})$. Let $\Gamma_{\text{out}}[T]$ be the drawing of T in Γ_{out}. By Theorem 7, we obtain an aligned drawing (Γ_{in}, L) with T drawn as $\Gamma_{\text{out}}[T]$. Moreover, since the drawing of T is fixed and is intersected only by line L, (Γ_{in}, A) is an aligned drawing. Thus, according to Lemma 6, there exists an aligned drawing of $(G, \mathcal{A})$.

If G_T does not contain separating triangles but contains either a free edge or a 0-anchored aligned edge e, let G_T/e be the graph after the contraction of e. Observe that, since $(G_T, \mathcal{A})$ is proper, every edge on the outer face is 1-crossed, and therefore every chord is ℓ-crossed, $\ell \geq 1$. Thus, e is an interior edge of $(G_T, \mathcal{A})$ and is not a chord. Therefore, by Lemma 7, $(G_T/e, \mathcal{A})$ is a proper aligned triangulation. By induction hypothesis, there exists an aligned drawing of $(G_T/e, \mathcal{A})$, and thus, by the same lemma, there exists an aligned drawing of $(G_T, \mathcal{A})$. □

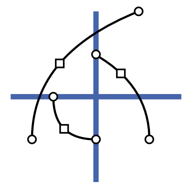


Figure 16: Placement of subdivision vertex to obtain a 2-aligned graph of alignment complexity $(1, 0, \perp)$.

Theorem 9 *Every* 2*-aligned graph has an aligned drawing with at most one bend per edge.*

Proof: We subdivide 2-crossed, 2-anchored or 1-crossed 1-anchored edges as depicted in Fig. 16. Thus, we obtain a 2-aligned graph $(G', \mathcal{A})$ of alignment complexity $(1, 0, \perp)$. Applying Theorem 8 to $(G', \mathcal{A})$ yields a one bend drawing of $(G, \mathcal{A})$. □

5 Conclusion

In this paper, we showed that if $\mathcal{A}$ is stretchable, then every k-aligned graph $(G, \mathcal{A})$ of alignment complexity $(1, 0, \perp)$ has a straight-line aligned drawing. As an intermediate result, we showed that a 1-aligned graph $(G, \mathcal{R})$ has an aligned drawing with a fixed convex drawing of the outer face. We showed that the less restricted version of this problem, where we are only given a set of vertices to be aligned, is $\mathcal{NP}$-hard but fixed-parameter tractable.

The case of more general alignment complexities is wide open; refer to Table 1. Our techniques imply the existence of one-bend aligned drawings of general 2-aligned graphs as Theorem 9 shows. However, the existence of straight-line aligned drawings are unknown even if in addition to 1-crossed edges, we only allow 2-anchored edges, i.e., in the case of alignment complexity $(1, 0, 0)$. In particular, there exist 2-aligned graphs that neither contain a free edge nor an aligned edge but their size is unbounded in the size of the arrangement; see Fig. 17. It seems that further reductions are necessary to arrive at a base case that can easily be drawn. This motivates the following questions.

1) What are all the combinations of line numbers k and alignment complexities C such that for every k-aligned graph $(G, \mathcal{A})$ of alignment complexity C there exists a straight-line aligned drawing provided $\mathcal{A}$ is stretchable?
2) Given a k-aligned graph $(G, \mathcal{A})$ and a line arrangement A homeomorphic to $\mathcal{A}$, what is the computational complexity of deciding whether $(G, \mathcal{A})$ admits a straight-line aligned drawing (Γ, A)?

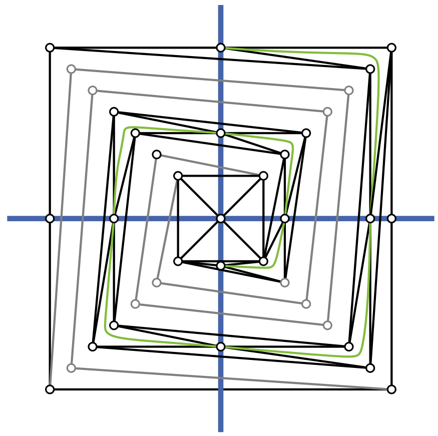


Figure 17: Sketch of a 2-aligned triangulation without aligned or free edges. The green edges are 2-anchored. The triangulation can be completed as indicated by the black edges.

Acknowledgements We thank the anonymous reviewers for thoroughly reading an earlier version of this paper and for providing useful comments.

References

[1] T. C. Biedl, M. Kaufmann, and P. Mutzel. Drawing Planar Partitions II: HH-Drawings. In J. Hromkovič and O. Sýkora, editors, *Proceedings of the 24th Workshop on Graph-Theoretic Concepts in Computer Science (WG'98)*, pages 124–136. Springer Berlin/Heidelberg, 1998. doi:10.1007/10692760_11.

[2] J. Cano, C. D. Tóth, and J. Urrutia. Upper Bound Constructions for Untangling Planar Geometric Graphs. *SIAM Journal on Discrete Mathematics*, 28(4):1935–1943, 2014. doi:10.1137/130924172.

[3] S. Chaplick, K. Fleszar, F. Lipp, A. Ravsky, O. Verbitsky, and A. Wolff. Drawing Graphs on Few Lines and Few Planes. In Y. Hu and M. Nöllenburg, editors, *Proceedings of the 24th International Symposium on Graph Drawing (GD'16)*, pages 166–180. Springer-Verlag Springer International Publishing, 2016. doi:10.1007/978-3-319-50106-2_14.

[4] S. Chaplick, K. Fleszar, F. Lipp, A. Ravsky, O. Verbitsky, and A. Wolff. The Complexity of Drawing Graphs on Few Lines and Few Planes. In F. Ellen, A. Kolokolova, and J.-R. Sack, editors, *Proceedings of the 15th International Symposium on Algorithms and Data Structures (WADS'17)*, pages 265–276. Springer-Verlag Springer International Publishing, 2017. doi:10.1007/978-3-319-62127-2_23.

[5] G. Da Lozzo, V. Dujmović, F. Frati, T. Mchedlidze, and V. Roselli. Drawing Planar Graphs with Many Collinear Vertices. In Y. Hu and M. Nöllenburg, editors, *Proceedings of the 24th International Symposium on Graph Drawing (GD'16)*, pages 152–165. Springer-Verlag Springer International Publishing, 2016. doi:10.1007/978-3-319-50106-2_13.

[6] V. Dujmović. The Utility of Untangling. *Journal of Graph Algorithms and Applications*, 21(1):121–134, 2017. doi:10.7155/jgaa.00407.

[7] V. Dujmović, W. Evans, S. Kobourov, G. Liotta, C. Weibel, and S. Wismath. On Graphs Supported by Line Sets. In U. Brandes and S. Cornelsen, editors, *Proceedings of the 18th International Symposium on Graph Drawing (GD'10)*, pages 177–182. Springer Berlin/Heidelberg, 2011. doi:10.1007/978-3-642-18469-7_16.

[8] V. Dujmović and S. Langerman. A Center Transversal Theorem for Hyperplanes and Applications to Graph Drawing. *Discrete & Computational Geometry*, 49(1), 2013. doi:10.1007/s00454-012-9464-y.

[9] U. Fößmeier and M. Kaufmann. Nice Drawings for Planar Bipartite Graphs. In G. Bongiovanni, D. P. Bovet, and G. D. Battista, editors, *Algorithms and Complexity*, pages 122–134. Springer Berlin/Heidelberg, 1997. doi:10.1007/3-540-62592-5_66.

[10] M. R. Garey, D. S. Johnson, and R. Tarjan. The Planar Hamiltonian Circuit Problem is NP-Complete. *SIAM Journal on Computing*, 5(4):704–714, 1976. doi:10.1137/0205049.

[11] J. E. Goodman and R. Pollack. Proof of Grünbaum's Conjecture on the Stretchability of Certain Arrangements of Pseudolines. *Journal of Combinatorial Theory, Series A*, 29(3):385 – 390, 1980. doi:10.1016/0097-3165(80)90038-2.

[12] F. Levi. Die Teilung der projektiven Ebene durch Geraden oder Pseudogerade. *Berichte Mathematische-Physikalischer Klassen der Sächsischen Akademie der Wissenschaften Leipzig*, 78:256–267, 1926.

[13] N. E. Mnëv. The Universality Theorems on the Classification Problem of Configuration Varieties and Convex Polytopes Varieties. In O. Y. Viro and A. M. Vershik, editors, *Topology and Geometry — Rohlin Seminar*, volume 1346 of *Lecture Notes in Computer Science*, pages 527–543. Springer Berlin/Heidelberg, 1988. doi:10.1007/BFb0082792.

[14] A. Ravsky and O. Verbitsky. On collinear sets in straight-line drawings. In P. Kolman and J. Kratochvíl, editors, *Proceedings of the 37th Workshop on Graph-Theoretic Concepts in Computer Science (WG'11)*, pages 295–306, 2011.

[15] P. Shor. Stretchability of Pseudolines is NP-Hard. In *Applied Geometry and Discrete Mathematics–The Victor Klee Festschrift*, volume 4 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pages 531–554. ACM Press, 1991.

[16] W. T. Tutte. How to Draw a Graph. *Proceedings of the London Mathematical Society*, 3(1):743–767, 1963. doi:10.1112/plms/s3-13.1.743.

[17] M. Wahlström. Abusing the Tutte Matrix: An Algebraic Instance Compression for the K-set-cycle Problem. In N. Portier and T. Wilke, editors, *Proceedings of the 30th International Symposium on Theoretical Aspects of Computer Science (STACS '13)*, pages 341–352. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2013. doi:10.4230/LIPIcs.STACS.2013.341.