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1-Planarity of Graphs with a Rotation System

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Abstract

A graph is 1-planar if it can be drawn in the plane such that each edge is crossed at most once. 1-planarity is known **NP**-hard, even for graphs of bounded bandwidth, pathwidth, or treewidth, and for near-planar graphs in which an edge is added to a planar graph. On the other hand, there is a linear time 1-planarity testing algorithm for maximal 1-planar graphs with a given rotation system.

In this work, we show that 1-planarity remains **NP**-hard even for 3-connected graphs with (or without) a rotation system. Moreover, the crossing number problem remains **NP**-hard for 3-connected 1-planar graphs with (or without) a rotation system.

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1 Introduction

Planar graphs have attracted researchers since the 1930's. There are numerous results on planar graphs such as forbidden minors, duality, efficient planarity tests, and straight line drawings, see [9, 17, 21]. More recently, researchers have investigated "beyond" planar graphs which generalize planar graphs by restrictions on crossings. A particular example is 1-planar graphs which can be drawn in the plane with at most one crossing per edge. 1-planar graphs were introduced by Ringel [24] and appear when a planar graph and its dual are drawn simultaneously. Here an edge and its dual cross.

1-planar graphs have recently received much interest. For every graph G there is a 1-planar graph which is obtained from G by subdividing edges. It was independently shown by several authors that a 1-planar graph with n vertices has at most $4n-8$ edges [4,6,11,22] and this upper bound is tight. However, there are 1-planar graphs with only $2.64n$ edges [7], in which any further edge destroys 1-planarity. 1-planar graphs are 6-colorable [5]. They do not admit straight-line drawings, which are excluded by so-called B- and W-configurations [25]. In the absence of these configuration there is a linear time algorithm to convert an embedded 1-planar graph into a straight-line drawing [16]. Also, 3-connected 1-planar graphs can be drawn straight-line on quadratic area with the exception of at most one edge in the outer face [1], whereas all 1-planar graphs admit a special bar 1-visibility representation on quadratic area, in which each vertical line of an edge crosses at most one horizontal bar of a vertex and vice versa [6].

1-planarity is **NP**-hard. This was proved first by Korzhik and Mohar [18] and improved to hold for graphs of bounded bandwidth, pathwidth, or treewidth by Bannister et al. [2], and for near planar graphs by Cabello and Mohar [8], where a near planar graph is obtained from a planar graph by the addition of one edge. The parameterized complexity of the problem with respect to several parameters was addressed in [2].

On the other hand, there is a linear time 1-planarity testing algorithm if the graphs are maximal 1-planar and are given with a rotation system by Eades et al. [10]. A graph is maximal 1-planar if the addition of an edge destroys 1-planarity. A rotation system describes the cyclic ordering of the edges at the vertices as obtained from a drawing. The algorithm of Eades et al. follows the edges in counter-clockwise order as given by the rotation system and decides that there is a planar face if there is a simple cycle around a face and a crossing if a vertex is traversed twice in a cycle.

A rotation system is the common output of a planarity test. It is used to compute planar embeddings and straight-line planar drawings in linear time [9, 17, 21]. A planar embedding partitions the plane into regions, called faces, and is specified by the cyclic ordering of the vertices and edges around each face. For planar graphs a rotation system and an embedding can be taken as synonyms, and one can be computed from the other in linear time. Rotation systems play a crucial role in this work.

A rotation system makes the essential difference to the complexity of upward planarity testing. A directed graph is upward planar if it can be drawn in the plane such that the curves of the edges are monotonically increasing in y-direction. Garg and Tamassia [14] showed that upward planarity testing of a graph is **NP**-hard. However, there is a linear time algorithm if a rotation system is given [3, 9]. In contrast, the **NP**-hard crossing number problem [13] remains **NP**-hard even with a given rotation system [23]. There is a parallel situation for 1-planarity.

We show that 1-planarity testing remains **NP**-hard even for 3-connected, 2-planar graphs with a given rotation system. Our **NP**-reduction is general enough to hold without a rotation system, and it can be modified to show that the crossing number problem remains **NP**-hard even for 1-planar graphs. Our proof is by reduction from the planar 3-SAT problem [20] and is simpler than the one by Korzhik and Mohar [18]. In our reduction, we introduce a "membrane technique" and directly encode the truth value of a literal by an edge crossing. Hence, given a rotation system, the borderline between tractable and intractable instances of 1-planarity is between maximal and 3-connected graphs.

The paper is organized as follows: We recall the basic notions in Section 2. Our main result - the **NP**-hardness of 1-planarity with a given rotation system - is presented in Section 3. In Section 4, we sharpen the **NP**-hardness result to 3-connected 2-planar graphs and improve upon the crossing number problem in Section 5. We conclude with some open problems in Section 6.

2 Preliminaries

We consider simple undirected graphs $G = (V, E)$ with n vertices and m edges. A *drawing* of a graph is a mapping of G into the plane such that the vertices are mapped to distinct points and each edge to a Jordan arc between its endpoints. A drawing is *plane* if (the Jordan arcs of) the edges do not cross and it is k-*plane* if each edge is crossed at most k times. In 1-plane drawings, crossings of edges with the same endpoint are excluded. A graph is *1-planar* if it admits a 1-planar drawing.

Each plane (1-plane) drawing of a graph implies a *rotation system*. The rotation at a vertex is the clockwise ordering of its incident edges as implied by the drawing. A rotation system consists of the rotations of all vertices. However, a given rotation system of a graph may not allow for a plane (1-plane) drawing, even if the graph is planar (1-planar), as K_4 (K_5) with all vertices in the outer face shows. We call a rotation system planar (1-planar) if it admits a plane (1-plane) drawing.

Similar to planar embeddings, a *1-planar embedding* specifies the faces in a 1-planar drawing. A face in a 1-planar embedding is given by a cyclic list of edges and edge segments, which occur in the case of a crossing. Then an edge consists of two segments. A 1-planar embedding uniquely implies a 1-planar rotation system. However, a 1-planar rotation system does not uniquely define a 1-planar embedding nor does it determine the pairs of crossing edges. In fact, we shall use this "gap" to show that deciding whether a rotation system is 1-planar is **NP**-hard.

Let G be a 1-planar embedded graph and denote by $G^\times$ its *planarization.* $G^\times$ is obtained from G by replacing each pair $e = \{u, v\}$ and $e' = \{u', v'\}$ of crossing edges by a new vertex of degree four joined to u, v, u', and v'. Then, $G^\times$ is a planar embedded graph, where its embedding is inherited from the embedding of G.

Finally, recall that the *planar 3-SAT problem* specializes the standard 3-SAT problem, such that the bipartite graph $(X \cup C, M)$ is planar, where $X = \{X_1, X_2, \ldots, X_i\}$ is the set of variables and $C = \{C_1, C_2, \ldots, C_j\}$ is the set of clauses and there is an edge $\{X_i, C_j\}$ if the variable X_i occurs as a positive or negative literal in C_j. Planar 3-SAT is known to be **NP**-hard [20].

3 NP-hardness of 1-Planarity Testing

In this section, we reduce planar 3-SAT to 1-planarity using gadgets for literals, variables and clauses. The key idea is a "membrane technique" and the encoding of truth values of variables by crossings. The membrane covering a clause C has five slots for the connection of its literals. Each connection forms a "Y" and crosses two edges of the membrane if the upper part is used and one edge if the lower part is used, respectively. The latter case corresponds to a satisfiable assignment of C.

The *U-graphs* from [18] are used as basic building blocks of our reduction, see Fig. 1 for an example. The vertices labeled $3, 2, 1, b, b-1, b-2$ are called *boundary vertices* and an edge connecting two boundary vertices is called *boundary edge*. Korzhik and Mohar [18] proved that a U-graph has a unique 1-planar embedding if it has at least $b \geq 6$ boundary vertices. In our reduction, we attach barrier edges and gadgets for variables (*V-gadgets*) and clauses (*C-gadgets*) to the boundary vertices and we assume that the number of boundary vertices is always at least 6 and sufficiently large for the case at hand.

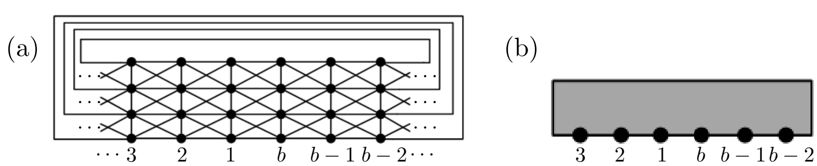

Figure 1: (a) The U-graph and (b) its abbreviation.

Let G be the planar embedded graph corresponding to a planar 3-SAT expression α. In the following, we construct a graph G_S^* endowed with a rotation system that is 1-planar if and only if α is satisfiable (see Fig. 2 for an example). The rotation system can be obtained directly from the given drawings. Let G^* be the dual graph of G. First, we transform G^* into a *U-supergraph* G_S^* as described in [19]. The construction replaces every vertex of G^* with a U-graph. Two adjacent vertices of G^* are connected in G_S^* by a set of l edges between distinct endpoints, called *barriers*, where we choose $l \geq 7$ for reasons which will be described later. A U-supergraph has a unique 1-planar embedding [18]. For every vertex v of G that represents a clause (resp. variable), we add a C-gadget (resp. V-gadget) to G^*_S. Let v be a vertex of G and f the corresponding face in G^*, and let F' be the set of vertices of G^* on the boundary of f. In G^*_S, the vertices in F' are replaced by U-graphs F'_U. In our construction, we attach each C- or V-gadget to an arbitrary U-graph in F'_U such that the gadget lies inside the face of G^*_S that corresponds to v in G. Fig. 2(b) shows an example in which V-gadget X_1 is attached to U-graph U_{f_1}. Finally, for every edge between a clause and a variable vertex in G, we add a path, called *rope*, between the corresponding C- and V-gadgets in G^*_S. For the number of edges of a rope, we choose two more edges than the number of edges of a barrier. As we will see later, a rope acts as a communication line that "passes" a crossing at a V-gadget to the C-gadgets at its other end. In fact, we construct a simultaneous embedding of G and its dual G^* by drawing our gadgets and the U-supergraph altogether.

A simple example for G^*_S is given in Fig. 2. The graph is obtained from a planar 3-SAT instance consisting of two clauses C_1, C_2 and three variables X_1, X_2, X_3 with the corresponding planar graph G, see Fig. 2(a). The vertices of G are depicted as circles and the edges as straight-line segments, the vertices of the dual graph G^* as squares and the edges as curled lines. Fig. 2(b) shows G^*_S, which is obtained from G and G^*. The shaded rectangles represent the U-graphs, which are connected by the barriers, drawn as a bundle of lines. The semi-ellipses are the C- and V-gadgets with the corresponding labels. The ropes are depicted as dashed lines. For the reasoning later on, we need that boundary edges in U-graphs are not crossed.

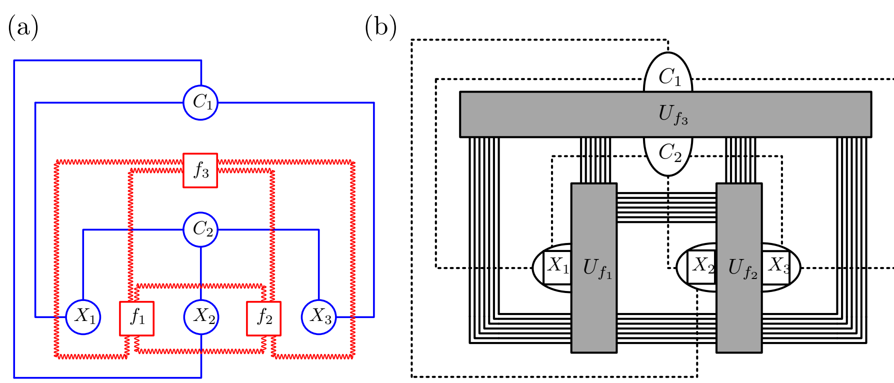

Figure 2: Example for the 1-planar graph constructed in the reduction.
(a) The plane drawing of a planar 3-SAT expression and its dual graph.
(b) The corresponding U-supergraph G^*_S with the clause and variable gadgets.

Lemma 1 *In a 1-planar drawing of G_S^* respecting the given rotation system, a boundary edge of a U-graph is never crossed.*

Since we need the structure of the C- and V-gadgets in order to prove Lemma 1, we postpone the proof until all the necessary definitions are made.

First, we consider C-gadgets used for the clauses, see Fig. 3 for an example. The gadget is attached to consecutive boundary vertices $b_1, \ldots, b_6$ of a U-graph. These vertices form the *clause base*. The vertices v_1, v_2, v_3 are the *variable vertices*, where each vertex corresponds to a literal in the clause. Hence, there are always three variable vertices. A variable vertex is connected to two vertices of the clause base by *anchor edges*. Additionally, a variable vertex is connected to the corresponding V-gadget via a rope. The edge from a variable vertex to the rope is called *variable edge* ($\{v_i, t_i\}$ for $i = 1, 2, 3$ in Fig. 3(a)). We introduce a path from b_1 to b_6, called *membrane*, which consists of the *membrane vertices* $m_1, \ldots, m_4$ connected by *membrane edges*.

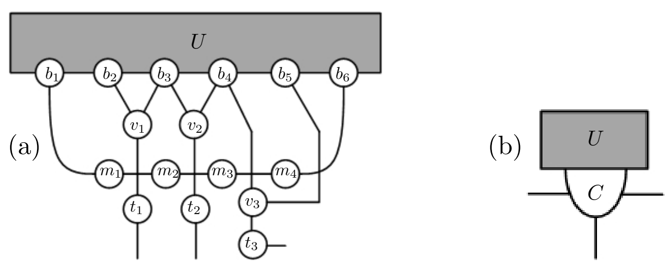

Figure 3: (a) A clause gadget and (b) its abbreviation.

In the following lemma, we need that the rope crosses at least one edge of a V-gadget. In Lemma 4, we will show that this precondition is always fulfilled.

Lemma 2 *In every 1-planar drawing of G_S^* respecting the given rotation system, at least one incident edge of each vertex v_1, v_2, v_3 of a C-gadget is crossed by a membrane edge if the rope crosses at least one edge of the attached V-gadget.*

Proof: Without loss of generality, we consider v_1. The first possibility to avoid a crossing of an adjacent edge of v_1 with a membrane edge is to cross a different edge of the rope other than $\{v_1, t_1\}$. A rope connects the C-gadget with a V-gadget with a barrier in between. In a 1-planar drawing, every edge of the rope must cross an edge of the barrier and an edge of the attached V-gadget by assumption. The size of a rope is the size of a barrier plus two. Hence, there is only one rope edge left to cross, namely, the variable edge $\{v_1, t_1\}$. Thus, to avoid a crossing of v_1's edges with the membrane, the membrane needs to be drawn "around" the whole rope of v_1, i. e., the face enclosed by the membrane and the clause base must include the whole rope. The membrane, consisting of five edges, must then be routed through at least one barrier, consisting of at least seven edges, which is impossible. □

From the proof of Lemma 2 we obtain that there are only two possibilities for a variable vertex v: (A) Both anchor edges of v are crossed by a membrane edge and its variable edge is not crossed by a membrane edge. (B) The variable edge of v is crossed by a membrane edge and none of its two anchor edges is crossed by a membrane edge. If (B) holds, we say that a variable vertex lies *inside*, as seen in Fig. 3(a), where vertices v_1 and v_2 lie inside. If (A) holds, a variable vertex lies *outside*, e.g., vertex v_3 lies outside in Fig. 3(a). As a direct consequence of (A) and (B), in a 1-planar drawing it is not possible that all three variable vertices of a C-gadget lie outside at the same time, as this would require six membrane edges to be crossed. We exploit this property to encode if a clause represented by the C-gadget is satisfied, i. e., it is satisfied if and only if at least one variable vertex lies inside, which holds if and only if a 1-planar drawing is possible.

The V-gadget for a variable consists of several *literal gadgets (L-gadgets)*, which are attached consecutively to the same U-graph. L-gadgets are similar to C-gadgets and come in two flavors, namely, a positive and a negative version, Fig. 4(a) depicts a positive L-gadget. The truth value of a single literal is encoded by a crossing of a certain edge of an L-gadget. An L-gadget has a *positive literal vertex* l^+ and a *negative literal vertex* l^- that are connected to three and two, respectively, consecutive boundary vertices of a U-graph via anchor edges. Vertex l^+ is connected to a rope vertex via a *clause edge*, named "Clause" in Fig. 4(a). Additionally, l^+ is adjacent to the negative literal vertex of a neighboring L-gadget of the same V-gadget ("In" in Fig. 4(a)). Similarly, vertex l^- is connected to the positive literal vertex of another neighboring L-gadget ("Out" in Fig. 4(a)). As in C-gadgets, boundary vertices b_1 and b_7 in an L-gadget are connected by a membrane, consisting of the membrane vertices m_1, m_2, m_3. In a negative L-gadget the clause edge is incident to the negative literal vertex l^- instead of the positive literal vertex l^+. Intuitively, the clause edge propagates the truth assignment of the literal via a rope to the clause in which the literal occurs. If the clause edge crosses the membrane, the literal is assigned false and the variable edge at the other end of the rope does not cross the membrane at the clause gadget. Otherwise, the literal is assigned true. The edges marked "In" and "Out" propagate the truth assignment of the literal to the other L-gadgets of the same variable to ensure a consistent truth value, i. e., either all positive or all negative L-gadgets cross their membranes. To ensure a consistent truth value, we additionally need the *terminal L-gadget* which is an L-gadget with no connection to a rope. The terminal L-gadgets are placed at the beginning and end of a series of L-gadgets.

Lemma 3 *In every 1-planar drawing of G_S^* respecting the given rotation system, at least one incident edge of each vertex l^+ and l^- of an L-gadget is crossed by a membrane edge.*

Proof: Vertex l^+ is adjacent to the l^- vertex of the neighboring L-gadget, now referred to as $\widehat{l^-}$. In order to avoid a crossing between any edge adjacent to l^+ and a membrane edge, the membrane has to be drawn such that it encloses

$\widehat{l^-}$. However, this is not possible, since then the edges from $\widehat{l^-}$ to the boundary vertices have to be crossed by the edges of two membranes. The argument for l^- is analogous. □

Similar to the variable vertices in C-gadgets, a literal vertex lies *inside* if its "In" edge is crossed by a membrane edge, which implies that its clause edge (if existent) is crossed by another membrane edge (l^+ in Fig. 4(c)). A literal vertex lies *outside* if all its anchor edges are crossed by membrane edges (l^+ in Fig. 4(a)).

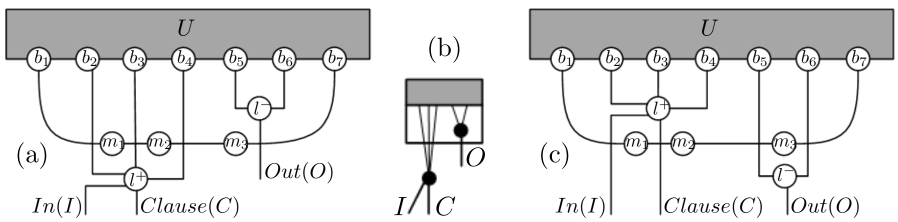

Figure 4: A 1-planar embedding of a positive literal gadget if the variable is (a) true or (c) false, respectively. (b) Abbreviation for "true" of a positive literal gadget.

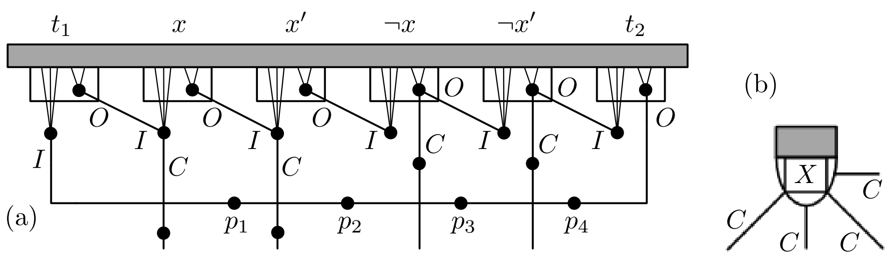

Figure 5: (a) A variable gadget consisting of two positive followed by two negative gadgets which are enclosed by two terminal gadgets. The value of the variable is true. (b) Its abbreviation.

Let X be a variable of a planar 3-SAT expression and let $v \in V$ be the vertex in G that corresponds to X. We construct the V-gadget of X as follows (see Fig. 5(a) for the result). The V-gadget is attached to a U-graph U that is adjacent to the face corresponding to v in G_S^*. First, attach a terminal gadget t_1 to U. Then, subsequently attach a positive or negative L-gadget depending on the occurrences of X according to a total order obtained from the rotation system of v, such that ropes attached to the V-gadget do not cross. Fig. 6 shows an example of how the ordering of L-gadgets in the V-gadget for variable X is obtained from the rotation system of the vertex representing X. Suppose the V-gadget is attached to U_2 (shown by the dotted semi-ellipses in Fig. 6), then the ordering of the L-gadgets from left to right is a, d, c, d. If it is attached to U_4 (shown by the dashed semi-ellipses in Fig. 6), then the ordering is d, a, b, c.

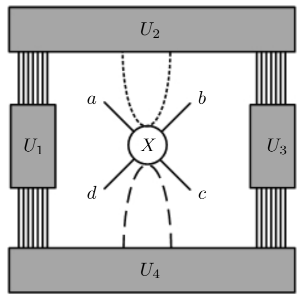

Figure 6: The rotation system of X determines the ordering of L-gadgets a, b, c, d of a V-gadget representing a variable X.

Intuitively, the planarity of G is preserved in G_S^*. Append the second terminal t_2. For each L- and terminal gadget, connect its "In" edge with the "Out" edge of its immediate neighbor, where "In" of t_1 is connected to "Out" of t_2 by a path, called *outer membrane*. The number of edges of the outer membrane is one above the number of occurrences of X.

Finally, we consider barriers and ropes. By the *size* of a barrier or rope we refer to the number of their edges. Korzhik and Mohar [18] proved that barriers of size at least 7 result in unique 1-planar embedded U-supergraphs. Let l be the maximum number of occurrences of a variable in the given SAT expression. For the size of the barriers we choose $\max\{7, l + 2\}$. Note that the size of the outer membrane of a V-gadget is one above the number of times the corresponding variable occurs in the expression. Consequently, an outer membrane has strictly fewer edges than a barrier. Thus, an outer membrane can never cross a barrier. For every edge of G, a rope connecting the V-gadget with the C-gadget is introduced in G_S^*. More precisely, a rope always connects one of the literal vertices of an L-gadget with one of the variable vertices of a C-gadget such that the planar rotation system of G is respected. The size of a rope is the size of a barrier plus 2. Figure 7 shows an example for a rope $r = \{l^+, r_1\}, \{r_1, r_2\}, \ldots, \{r_8, v\}$ of size 9. A rope crosses each edge of a barrier exactly once. Then, the rope crosses either two edges of a V-gadget (Fig. 7(b)) or one edge of a V-gadget and one edge of a C-gadget (Fig. 7(a)). Crossings of the first and last edge of a rope (e. g., $\{l^+, r_1\}, \{r_8, v\}$ of r) propagate the truth assignment of the literal at the one end to the clause at the other end. Consider again Fig. 7(a), where the positive L-gadget x, belonging to the V-gadget of variable X, is connected to C-gadget C. In the figure, X is assigned true as l^+ lies outside. The rope propagates the truth assignment to the clause, where variable vertex x can then lie inside and, hence, the clause is satisfied by X. Consider now Fig. 7(b), where the variable X is assigned false and, hence, l^+

lies inside. Consequently, edge $\{l^+, r_1\}$ is crossed by the membrane of x and edge $\{r_1, r_2\}$ is crossed by the outer membrane $\{p, p'\}$. Hence, every remaining edge of r is crossed by the barrier and, therefore, the variable vertex of C must lie outside, representing that the clause is not satisfied by x. Now suppose that every literal of the clause is assigned false. Thus, all three variable vertices of C lie outside. However, then there is no 1-planar drawing of C and, hence, no 1-planar drawing of G_S^*. Hence, C and thus the whole 3-SAT expression is not satisfiable.

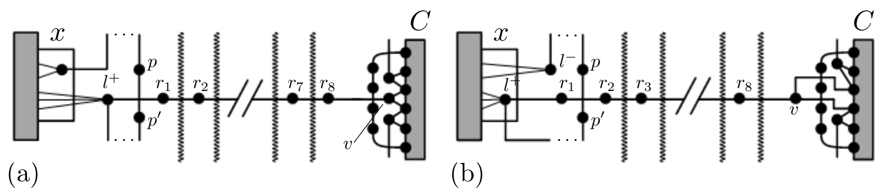

Figure 7: From left to right: Fragment of a variable gadget that shows a literal gadget and a part of its outer membrane, a barrier drawn as curled lines, and a clause gadget. (a) The literal x is true, hence the clause represented by the C-gadget on the right is satisfied by v. (b) The literal x is false, hence the clause is not satisfied by v.

We are now ready to prove Lemma 1:

Proof: Denote by f a triangular face adjacent to a boundary edge of G_S^*. The lemma holds for the U-supergraph, i. e., G_S^* without C- and V-gadgets, and ropes as the U-supergraph has a unique 1-planar embedding. Let f be a triangular face adjacent to a boundary edge. Due to the unique embedding, no U-graph of the U-supergraph can be drawn inside f (cf. Fig. 1(a)). The same also holds for every vertex v of a rope, a C-, or a V-gadget, as in each case v has at least degree two. If v lies inside f, it would cause at least two crossings of a boundary edge. Consequently, a boundary edge can only be crossed if a whole rope, C-, or V-gadget lies inside f. In the case of C- and V-gadgets, this is impossible even for the membrane or outer membrane of V-gadgets, as the rotation system forces the first and last edges of the membrane to leave one of its endpoints outside f. These two edges alone would already cause two crossings of the boundary edge. Similarly, as a rope connects a V-gadget with a C-gadget, it cannot be drawn inside f. □

Before we can prove the main theorem, we need two additional lemmata.

Lemma 4 *Let x be an L-gadget of a V-gadget X. Then, in every 1-planar drawing of G_S^* respecting the given rotation system, the rope attached to x is crossed by the outer membrane of X.*

Proof: In order to avoid a crossing of the rope, the outer membrane of X has to be drawn "around" the C-gadget that is connected to x, i. e., the outer membrane encloses the C-gadget. However, then the outer membrane needs to cross at least one barrier, which is impossible since the size of the outer membrane is less than the size of a barrier. □

Lemma 5 *Let X be a V-gadget. In every 1-planar drawing of G^*_S respecting the given rotation system, all positive literal vertices l^+ of X's L-gadgets lie inside if and only if all negative literal vertices l^- of X's L-gadgets lie outside.*

Proof: Let x_1 and x_2 be any positive or negative L-gadget as part of X and let l^+_1, l^-_1 (l^+_2, l^-_2) be the literal vertices of x_1 (x_2). By Lemma 3, each of these literal vertices lies either inside or outside. It is not possible that both a positive and a negative literal vertex lie outside since a membrane of an L-gadget has size 4, whereas the literal vertices have a total of 5 anchor edges. Now suppose for contradiction that both l^+_1 and l^-_2 lie outside. As l^+_1 lies outside, l^-_1 lies inside and, consequently, the "Out" edge of l^-_1 is crossed by the membrane of x_1. Let l^+_3 be the positive literal vertex connected to l^-_1 via its "Out" edge. Vertex l^+_3 is "tugged" outside, i. e., l^+_3 cannot lie inside as its "In" edge (which is the same edge as the "Out" edge of l^-_1) is already crossed. If $l^+_3 = l^+_2$, then l^+_2 lies outside and, hence, l^-_2 must lie inside, a contradiction. Otherwise, l^+_3 belongs either to a terminal gadget or to another L-gadget. If l^+_3 belongs to a terminal gadget t, then the negative literal vertex l^-_3 of t must lie inside. Via the outer membrane, the information that l^-_3 lies inside is propagated to the other terminal gadget t' by the same mechanism that governs the ropes. Hence, the negative literal vertex of t' lies inside and the positive one lies outside. If l^+_3 belongs to another L-gadget, then also the negative literal vertex lies inside and the positive one lies outside. By subsequently applying these arguments, we eventually arrive at x_2 and can conclude that l^-_2 must also lie inside, a contradiction. The reasoning is analogous if l^-_1 and l^+_2 lie outside. □

Theorem 1 *1-planarity is* **NP**-*hard for a graph with a given rotation system.*

Proof: A planar 3-SAT expression α is satisfiable if and only if the graph G^*_S obtained from α is 1-planar.
"⇒": Draw the V-gadgets according to a satisfying truth assignment of the variables, i. e., the positive literal vertices of a variable gadget lie outside if and only if the corresponding variable is assigned true. Then, every C-gadget has a variable vertex that can lie inside and, thus, has a 1-planar drawing due to (A) and (B).
"⇐": We obtain a truth assignment of the variables from a 1-planar drawing of G^*_S as follows. A variable is assigned true if and only if the positive literal vertices of the corresponding V-gadget lie outside. The so obtained assignment is consistent by Lemma 5. In each C-gadget, at least one variable vertex lies inside. This vertex is connected, via a rope, to a literal vertex of a V-gadget which necessarily lies outside. Thus, the corresponding variable satisfies the clause at hand. Hence, the obtained truth assignment satisfies the 3-SAT expression. □

The reduction in the proof of Theorem 1 does not need the rotation system. The U-graphs have a unique 1-planar embedding and thus a fixed rotation system, and the rotation system of the C- and V-gadgets is fixed. Therefore, a 1-planar drawing of G^*_S would imply the used rotation system, and our construction for the **NP**-hardness proof goes through without the rotation system. Thus, we have an alternative proof to Theorem 5 in [18]. The only variable part comes from the planar embeddings of G. Each embedding uniquely defines its dual G^*, and conversely. The graph G^*_S is constructed from G^* and conversely G^*_S implies G^* and fixes the embedding and the rotation system.

Note that the graph G^*_S is 2-planar independent of the satisfiablity of the given 3-SAT expression. It suffices to stretch each rope to cross one more edge.

Corollary 1 *It is* **NP***-hard to decide whether a graph G is 1-planar, even if G is 2-planar.*

In contrast, 1-planarity is solvable in linear time for embedded graphs. Given an embedding of a graph G we first check whether an edge occurs in more than two faces. Then, we compute the planarization $G^\times$ of G and check its planarity.

4 Triconnected Graphs

We wish to push the **NP**-hardness of 1-planarity even further and increase connectivity. For this, we use the reduction from Section 3 and expand the constructed graph so that it is 3-connected. Note that the graphs in the **NP**-hardness proofs in [18] and [2] are at most 2-connected. The graph G^*_S consists of a U-subgraph G^*_U and of C- and V-gadgets and ropes. G^*_U is an expansion of the dual graph G^* whose vertices are U-graphs and adjacent vertices are connected by barriers consisting of $l \geq 7$ edges, which end at different boundary vertices. As the dual G^* is connected and each U-graph is 5-connected [18], G^*_U is 5-connected. The breaking points of G^*_S are the gadgets and ropes, which consist of paths (with a branching) between U-graphs. Hence, G^*_S is (only) 2-connected and not 3-connected, since membranes in clause gadgets have split pairs.

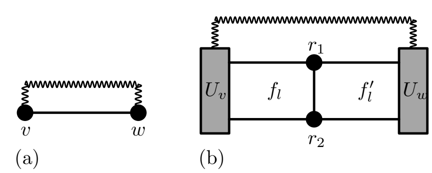

Figure 8: (a) A 2-connected graph is transformed into (b) a 3-connected graph by replacing the vertices by U-graphs and the edge by a ladder. The wave line is a path from v to w.

For 3-connectivity, we expand the C-and V-gadgets and the ropes of G^*_S and obtain the graph G_3. Let V^* and E^* be the set of vertices and edges of G^*_S, respectively, that are not part of a U-graph. In a nutshell, every vertex of V^*

is replaced by a U-graph with at least six boundary vertices and every edge of E^* is replaced by a gadget called *ladder*. Let $e = \{v, w\}$ be an edge in E^*, see Fig. 8(a). In G_S^*, there is a path (wave lines) between v and w which does not contain e. Replace v and w by U-graphs U_v and U_w and replace e by a ladder consisting of two *rung* vertices r_1, r_2 and five *ladder edges*, see Fig. 8(b).
The two faces enclosed by the ladder edges and the boundary edges of U_v and U_w denoted by f_l and f_l' are called *ladder faces*. Note that the ladder is connected to two distinct boundary vertices of U_v and U_w, respectively. Further, we need to make sure that G_3 respects the rotation system of G_S^*. Let $v \in V^*$ be a vertex in G_S^*. By construction, the minimum degree of v is two and the maximum degree is five. We replace v by a U-graph U_v with six boundary vertices. Every edge incident to v is replaced by a ladder that is connected to U_v such that the cyclic order of the ladders around U_v in G_3 respects the rotation system of v in G_S^* as shown in Fig. 9. In G_S^*, every C- and V-gadget, and every barrier is attached to a U-graph U. As the edges that connect the gadget with U are replaced by ladders, we add one additional boundary vertex to U for every ladder to which it is connected.

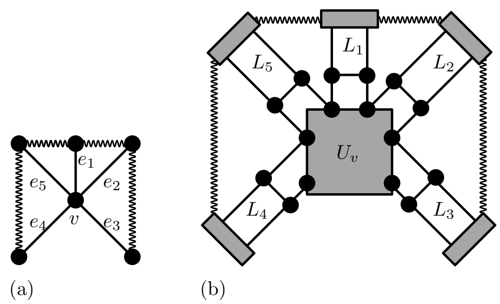

Figure 9: A vertex of degree 5 and its incident edges are replaced by a U-graph and 5 ladders. The wave lines are paths chosen arbitrarily to represent that the graph in (a) is 2-connected but not 3-connected, e. g., both endpoints of e_5 form a split pair. The graph in (b) is 3-connected.

As an example, Fig. 10 shows an L-gadget after the transformation. We obtain G_3 by replacing all gadgets analogously, and so remove all break points.

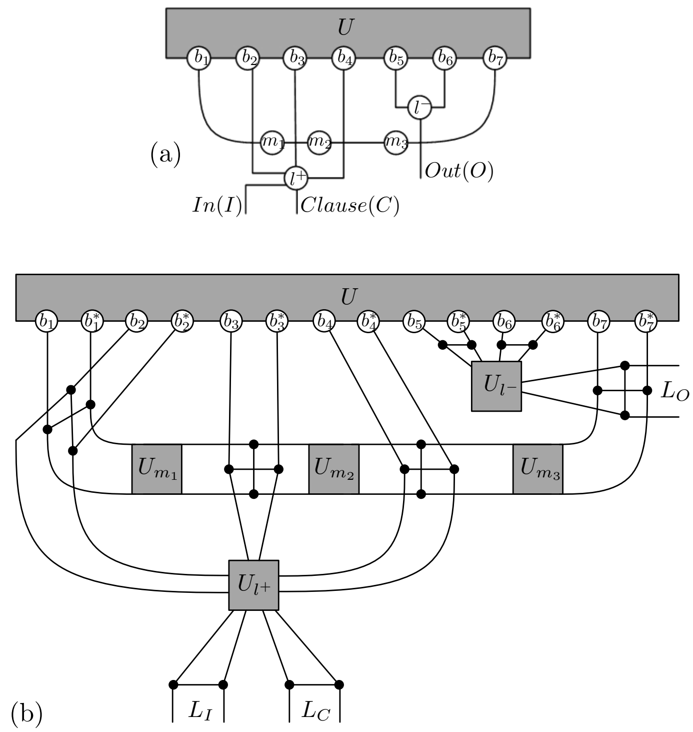

Figure 10: A 1-planar embedding of (a) a positive literal gadget if the variable is true and (b) the positive literal gadget transformed into a 3-connected graph.

Corollary 2 *The graph G_3 obtained from G^*_S is 3-connected.*

What is left to show is that G^*_S is 1-planar if and only if G_3 is 1-planar. If G^*_S has a 1-planar drawing, we can replace G^*_S by G_3 where two crossing edges that do not belong to a U-graph are replaced by two "crossing ladders" as shown in Fig. 12(b). By this, we obtain:

Corollary 3 *The rotation system of G_3 is 1-planar if the rotation system of G^*_S is 1-planar.*

For the converse, we need to examine other properties of G_3.

Lemma 6 *In a 1-planar drawing of G_3 respecting the given rotation system, the ladder edges belonging to a ladder never cross each other.*

Proof: As the rotation system is fixed, Fig. 11(b) shows the only way a ladder can be drawn such that ladder edges cross (apart from the symmetric case where U_w is "twisted"). U_v is enclosed by face f in Fig. 11(b). As G_3 is 3-connected there is a path p from U_v to U_w that is edge disjoint to the ladder.

By construction, p is neither incident to b_3 nor to b_4. Thus, p starts within f and ends outside f, which is not possible in a 1-planar drawing as the ladder edges are already crossed. The symmetric case with U_w is analogous. □

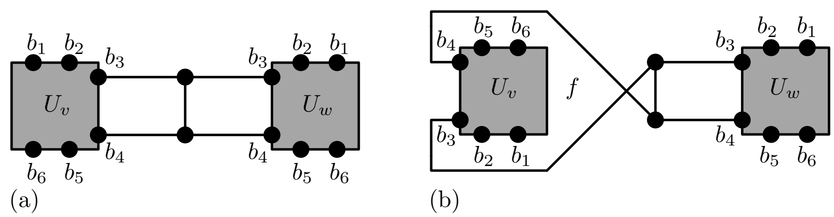

Figure 11: A 3-connected graph consisting of two U-graphs and a ladder. Note that the third path from U_v to U_w is omitted to avoid cluttering the image. The graphs in (a) and (b) have the same rotation system.

Lemma 6 ensures that f_l and f'_l (see Fig. 8(b)) are well defined, i. e., they always exist and are enclosed by ladder edges.

Lemma 7 *Assume that G_3 has a 1-planar drawing respecting the given rotation system. Let f_l and f'_l be the two ladder faces of ladder L. If a vertex v lies inside f_l or f'_l, v can only be a rung vertex of another ladder.*

Proof: Without loss of generality, let v be the vertex that lies inside ladder face f_l and let U_v be the U-graph adjacent to f_l. Suppose for contradiction that v is not a rung vertex. As in the construction of G_3, every vertex was replaced by a U-graph, v can only be the boundary vertex of a U-graph U. As no edge between two boundary vertices can be crossed (Lemma 1), U must completely lie inside f_l. Remember that U corresponds to a vertex w in G^*_S, in which every vertex and in particular w has at least degree two. Every edge incident to w has been replaced by a ladder and thus there are at least four edges connected to U in G_3. Now assume that only U and no other U-graph lies inside f_l. Face f_l is bounded by three ladder edges and boundary edges of U, the latter of which are not crossed by Lemma 1. Hence, the four edges connected to U must cross the three ladder edges of L, a contradiction. If other U-graphs besides U are within f_l a similar reasoning also leads to a contradiction. □

Lemma 8 *Let L_1 (L_2) be a ladder with ladder faces f_{l1}, f'_{l1} (f_{l2}, f'_{l2}) and rung vertices r_1, r'_1 (r_2, r'_2). If r_1 lies inside f_{l2} then r'_1 must lie inside f'_{l2}, and r_2 and r'_2 lie inside f_{l1} and f'_{l1}, respectively. The symmetric case where r_1 lies inside f'_{l2} holds analogously.*

Proof: Let U_{l2} be the U-graph adjacent to f_{l2}. For contradiction, suppose that r_2 lies outside f'_{l2}. r_1 has degree three and all edges incident to r_1 end outside f_{l2} by Lemma 7. Face f_{l2} is bounded by three ladder edges and boundary edges of U_{l2}, the latter of which must not be crossed by Lemma 1. Thus, in a 1-planar drawing one of the ladder edges e incident to r_1 must cross edge $\{r_2, r_2'\}$. By assumption, $e \neq \{r_1, r_2\}$. Let U_{l1} be the U-graph connected to e. Consequently, U_{l1} lies within f_{l2}', a contradiction to Lemma 6. Hence, edge $\{r_1, r_2\}$ must cross $\{r_2', r_2\}$, and thus r_2 lies within f_{l2}'. Analogously, r_2 and r_2' lie within f_{l1} and f_{l1}', respectively. □

From Lemma 6, 7 and 8 we obtain:

Corollary 4 *Let L be a ladder. In a 1-planar drawing of G_3 respecting the given rotation system, either no ladder edge of L is crossed or all of its ladder edges are crossed by ladder edges from another single ladder.*

The two possible 1-planar drawings of a ladder are depicted in Figures 8(b) and 12(b). We are now ready to conclude our proof.

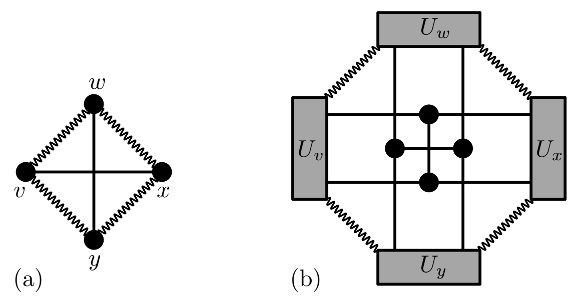

Figure 12: A crossing in (a) a 2-connected graph is replaced by (b) a 3-connected graph.

Lemma 9 *The rotation system of G_S^* is 1-planar if and only if the rotation system of G_3 is 1-planar.*

Proof:
"⇒": Follows from Corollary 3.
"⇐": In the 1-planar drawing of G_3, there are two types of crossing: the ones of the K_4s in the U-graphs and, by Cor. 4, the crossings between two ladders. By inverting the construction of G_3, we obtain a 1-planar drawing of G_S^* where two "crossing ladders" are replaced by the two corresponding crossing edges. □

Hence, 3-connectivity does not change the hardness of 1-planarity.

Theorem 2 *1-planarity is* **NP***-hard even if the graph is 3-connected and is given with a rotation system.*

As before, our construction does not depend on the rotation system and the graphs are 2-planar.

Corollary 5 *It is* **NP***-hard to decide whether a 3-connected graph G is 1-planar, even if G is 2-planar.*

5 Crossing Number

In this section, we use our membrane technique for an improvement on the crossing number problem to 1-planar graphs with a given rotation system. The crossing number problem asks whether there is a drawing of a graph in the plane with at most k edge crossings. The **NP**-hardness was first proved by Garey and Johnson [13] using graphs with parallel edges and vertices of high degree. Hliněný [15] improved the **NP**-hardness result to simple cubic graphs, and Pelsmajer et al. [23] showed that it remains **NP**-hard for a rotation system [23].

Theorem 3 *Crossing number is* **NP***-hard for 1-planar graphs with a given rotation system.*

Proof: We reduce from the **NP**-complete planar vertex cover problem [12]. Its input is a planar graph $G = (V, E)$ and a non-negative integer k and it asks whether there is a subset $V' \subseteq V$ with $|V'| \leq k$ such that every edge of G is incident to at least one vertex in V'. V' is then called a *vertex cover* of G.

Fix any planar embedding of G. As in Section 3, we start by transforming G into a U-supergraph G^*_S. Again, each pair of U-graphs corresponding to adjacent vertices of G^* is connected by $l = 7$ *barrier* edges. In the following, we describe the single type of a gadget, which is attached to the U-graphs of G^*_S for every vertex of G in the same way as C- and V-gadgets were attached in the reduction of Sect. 3. Consider $v \in V$ of degree d. See Fig. 13 for an example of the gadget in the case of $d = 3$. The gadget for v is attached to boundary vertices $b_1, \ldots, b_{d+3}$ of a U-graph and contains the *membrane vertices* $m_1, \ldots, m_d$ and the *connector* vertex c_v. There is a path, called *membrane*, from b_1 via $m_1, \ldots, m_d$ to b_{d+2}. We introduce $d + 1$ *anchor* edges connecting c_v to the vertices $b_2, \ldots, b_{d+2}$. For every edge $\{u, v\} \in E$ we add a path from c_u to c_v consisting of exactly $l + 1$ edges, called *rope*. We can choose the rotation system at c_v according to the embedding of G such that the ropes do not cross. In general, a gadget has two possible 1-planar embeddings, where c_v is placed *inside*, i. e., the membrane is crossed by the d ropes, or *outside*, i. e., the membrane is crossed by the $d + 1$ anchor edges. Note that the latter case yields one more crossing. Given a 1-planar embedding of G^*_S, we define the set $V' \subseteq V$ containing exactly those vertices v, whose corresponding connector vertices c_v are placed outside. For every edge $\{u, v\} \in E$, the rope connecting c_u with c_v has to cross l barrier edges. As it consists of only $l + 1$ edges, it has only one edge left for crossing a membrane. Thus, at least one of the vertices c_u or c_v must be placed outside and V' is a vertex cover of G. As each U-graph has a unique 1-planar embedding, let C be the constant number of crossings they contain. The total number of crossings of all membranes is $\sum_{v \in V} \deg(v) = 2m$ plus the number of connector vertices placed outside. Additionally, each of the $l \cdot m$ barrier edges are crossed by a rope. Now let $k' = C + (2 + l)m + k$.

If there is a 1-planar embedding of G^*_S with at most k' crossings, at most k connector vertices can be placed outside, i. e., V' is a vertex cover with $|V'| \leq k$.

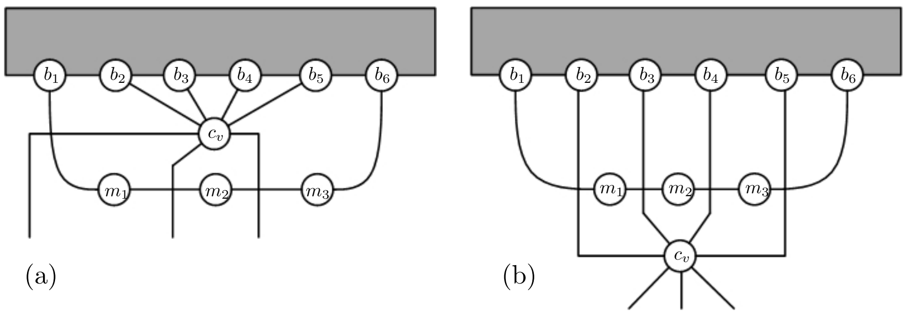

Figure 13: The gadget for proving that crossing number is **NP**-hard for 1-planar graphs.

Conversely suppose there is a vertex cover V' with $|V'| \leq k$. Specify the 1-planar embedding by placing exactly those connector vertices c_v outside with $v \in V'$. Then the resulting embedding of G_S^* has exactly $C + (2+l)m + k$ crossings. □

By using the construction from this chapter and adapting the number of crossings in the proof of Theorem 3, we can extend Theorem 3.

Theorem 4 *Crossing number is* **NP***-hard for 1-planar graphs with a given rotation system, even if the graphs are 3-connected.*

6 Conclusion

We have improved upon the **NP**-hardness of 1-planarity and added a rotation system, 3-connectivity and 2-planarity. The known tractable case treats maximal 1-planar graphs with a rotation system [10]. Maximal 1-planar graphs are 2-connected, and may contain vertices of degree two [7]. Are there other tractable instances, e.g., with a given rotation system and k-connectivity for $k = 4, 5, 6$? Is maximal 1-planarity testing tractable if the input is a graph? For our proofs, we introduced a "membrane" and "ladder" technique which may be useful in **NP**-hardness proofs, e. g., for k-planarity.

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References

[1] M. J. Alam, F. J. Brandenburg, and S. G. Kobourov. Straight-line drawings of 3-connected 1-planar graphs. In S. Wismath and A. Wolff, editors, *GD 2013*, volume 8242 of *LNCS*, pages 83–94. Springer, 2013. doi:10.1007/978-3-319-03841-4_8.

[2] M. J. Bannister, S. Cabello, and D. Eppstein. Parameterized complexity of 1-planarity. In F. Dehne, R. Solis-Oba, and J.-R. Sack, editors, *WADS 2013*, volume 8037 of *LNCS*, pages 97–108. Springer, 2013. doi:10.1007/978-3-642-40104-6_9.

[3] P. Bertolazzi, G. Di Battista, G. Liotta, and C. Mannino. Upward drawings of triconnected digraphs. *Algorithmica*, 12(6):476–497, 1994. doi:10.1007/bf01188716.

[4] R. Bodendiek, H. Schumacher, and K. Wagner. Über 1-optimale Graphen. *Mathematische Nachrichten*, 117:323–339, 1984. doi:10.1002/mana.3211170125.

[5] O. V. Borodin. A new proof of the 6 color theorem. *Journal of Graph Theory*, 19(4):507–521, 1995. doi:10.1002/jgt.3190190406.

[6] F. J. Brandenburg. 1-visibility representations of 1-planar graphs. *J. Graph Algorithms Appl.*, 18(3):421–438, 2014. doi:10.7155/jgaa.00330.

[7] F. J. Brandenburg, D. Eppstein, A. Gleißner, M. T. Goodrich, K. Hanauer, and J. Reislhuber. On the density of maximal 1-planar graphs. In W. Didimo and M. Patrignani, editors, *GD 2012*, volume 7704 of *LNCS*, pages 327–338, 2013. doi:10.1007/978-3-642-36763-2_29.

[8] S. Cabello and B. Mohar. Adding one edge to planar graphs makes crossing number and 1-planarity hard. *SIAM J. Comput.*, 42(5):1803–1829, 2013. doi:10.1137/120872310.

[9] G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis. *Graph Drawing: Algorithms for the Visualization of Graphs.* Prentice-Hall, 1999.

[10] P. Eades, S.-H. Hong, N. Katoh, G. Liotta, P. Schweitzer, and Y. Suzuki. A linear time algorithm for testing maximal 1-planarity of graphs with a rotation system. *Theor. Comput. Sci.*, 513:65–76, 2013. doi:10.1016/j.tcs.2013.09.029.

[11] I. Fabrici and T. Madaras. The structure of 1-planar graphs. *Discrete Mathematics*, 307(7–8):854 – 865, 2007. doi:10.1016/j.disc.2005.11.056.

[12] M. R. Garey and D. S. Johnson. The rectilinear Steiner tree problem is NP-complete. *SIAM J. Appl. Math.*, 32(4):826–834, 1977. doi:10.1137/0132071.

[13] M. R. Garey and D. S. Johnson. Crossing number is NP-complete. *SIAM J. Matrix Anal. A.*, 4(3):312–316, 1983. doi:10.1137/0604033.

[14] A. Garg and R. Tamassia. On the computational complexity of upward and rectilinear planarity testing. *SIAM J. Comput.*, 31(2):601–625, 2002. doi:10.1137/s0097539794277123.

[15] P. Hliněný. Crossing number is hard for cubic graphs. *J. Comb. Theory Ser. B*, 96(4):455–471, July 2006. doi:10.1016/j.jctb.2005.09.009.

[16] S.-H. Hong, P. Eades, G. Liotta, and S.-H. Poon. Fáry's theorem for 1-planar graphs. In J. Gudmundsson, J. Mestre, and T. Viglas, editors, *COCOON 2012*, volume 7434 of *LNCS*, pages 335–346. Springer, 2012. doi:10.1007/978-3-642-32241-9_29.

[17] M. Kaufmann and D. Wagner. *Drawing Graphs*, volume 2025 of *LNCS*. Springer, 2001. doi:10.1007/3-540-44969-8.

[18] V. Korzhik and B. Mohar. Minimal obstructions for 1-immersions and hardness of 1-planarity testing. *Journal of Graph Theory*, 72(1):30–71, 2013. doi:10.1002/jgt.21630.

[19] V. P. Korzhik. Minimal non-1-planar graphs. *Discrete Math.*, 308(7):1319–1327, 2008. doi:10.1016/j.disc.2007.04.009.

[20] D. Lichtenstein. Planar formulae and their uses. *SIAM J. Comput.*, 11(2):329–343, 1982. doi:10.1137/0211025.

[21] T. Nishizeki and S. Rahman. *Planar Graph Drawing*. World Scientific, 2004. doi:10.1142/5648.

[22] J. Pach and G. Tóth. Graphs drawn with few crossings per edge. *Combinatorica*, 17:427–439, 1997. doi:10.1007/bf01215922.

[23] M. J. Pelsmajer, M. Schaefer, and D. Štefankovič. Crossing number of graphs with rotation systems. *Algorithmica*, 30:679–702, 2011. doi:10.1007/978-3-540-77537-9_3.

[24] G. Ringel. Ein Sechsfarbenproblem auf der Kugel. *Abh. Math. Semin. Univ. Hamb*, 29:107–117, 1965. doi:10.1007/bf02996313.

[25] C. Thomassen. Rectilinear drawings of graphs. *J. Graph Theor.*, 12(3):335–341, 1988. doi:10.1002/jgt.3190120306.