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Multilayer Drawings of Clustered Graphs

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Abstract

The *cluster adjacency graph* of a flat clustered graph $C(G,T)$ is the graph A whose vertices are the clusters in T and whose edges connect clusters containing vertices that are adjacent in G. A *multilayer drawing* of a clustered graph C consists of a straight-line c-planar drawing of C in which the clusters are drawn as convex regions and of a straight-line planar drawing of A such that each vertex $a \in A$ is drawn in the cluster corresponding to a and such that no edge $(a_1, a_2) \in A$ intersects any cluster different from a_1 and a_2. In this paper, we show that every c-planar flat clustered graph admits a multilayer drawing.

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1 Introduction

A *clustered graph* is a pair $C(G,T)$, where G is a graph, called *underlying graph*, and T is a rooted tree, called *inclusion tree*, whose leaves are the vertices of G. Each internal node ν of T corresponds to the subset of vertices of G, called *cluster*, that are the leaves of the subtree of T rooted at ν. Throughout the paper, we assume that each path from the root of T to any leaf has the same number of edges, which is denoted by $h(T)$ (the motivation for such an assumption will be provided later in this section). We call *level* of a cluster μ the minimum number of edges in a path in T from μ to a leaf. Given a clustered graph $C(G,T)$, the *cluster adjacency graph at level* i is the graph A_i whose vertices are the clusters at level i and having an edge between two clusters μ and ν if any vertex in μ and any vertex in ν are connected by an edge of G. A clustered graph is *flat* if the height of T is at most two, i.e., no cluster different from the root contains other clusters. In a flat clustered graph $C(G,T)$, we say that the *cluster of* a vertex v of G is its parent in T and we denote it by $\mu(v)$; also, in a flat clustered graph we call *clusters* only the children of the root, and we call *adjacency graph* A the adjacency graph at level one; further, for any cluster μ in T, we denote by $a(\mu)$ the vertex representing μ in A.

Clustered graphs find applications in several areas of computer science and hence they have been widely studied from a theoretical point of view. Several methods have been developed to compute a *good clustering* for a given graph G, that is, for constructing a (usually flat) clustered graph that has G as underlying graph and that has high edge density inside each cluster and few edges connecting vertices belonging to different clusters. See [24] for a survey on graph clustering. On the other hand, in a typical graph drawing problem, the clustering is given as part of the input, and the goal is to visualize the clustered graph in a *readable way*.

Clustered planarity (also called c-planarity for short) is a concept, introduced in [15], that generalizes planarity to clustered graphs and that has been widely recognized as *the* standard for readability of clustered graph drawings. A drawing of a clustered graph represents each cluster μ as a closed simple region δ_μ of the plane containing all and only the vertices of μ; a drawing is *c-planar* if it contains no edge crossings (i.e., the drawing of the underlying graph is planar), no edge-region crossings (i.e., no edge intersects the border of a region δ_μ more than once), and no region-region crossings (i.e., each two regions δ_μ and δ_ν representing clusters μ and ν are disjoint). Figure 1 shows a c-planar drawing of a clustered graph. A graph is c-planar if it admits a c-planar drawing. Designing an algorithm to test whether a clustered graph is c-planar (or prove that no efficient algorithm exists) is one of the most studied and still far from solved graph drawing problems (see [2, 3, 4, 5, 6, 8, 17, 18, 19, 20, 21]). Assuming that a clustered graph C is c-planar, several algorithms and bounds are known for constructing c-planar drawings of C [7, 11, 13, 23]. A particular attention has been devoted to *straight-line convex drawings*, that are c-planar drawings requiring edges to be straight-line segments and clusters to be convex regions. Every c-planar graph admits a straight-line convex drawing [10], even if the shape of each cluster is fixed in advance [1]. Straight-line convex drawings might require exponential area [14].

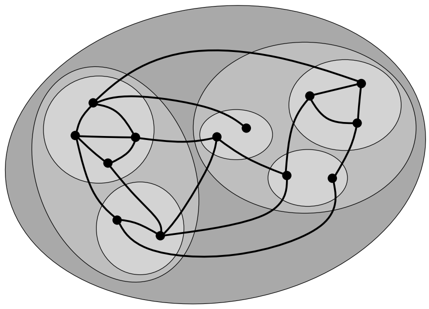


Figure 1: A c-planar drawing of a clustered graph.

In this paper we introduce and study *multilayer drawings* of clustered graphs. For a subset δ of the plane $z = 0$, we denote by δ^i the vertical projection of δ on the plane $z = i$.

Definition 1 *A* multilayer drawing *of a clustered graph $C(G, T)$ consists of a convex straight-line drawing Γ of C in the plane $z = 0$ (the* base layer*) and, for every $1 \leqslant i \leqslant h(T)$, of a straight-line planar drawing $\Gamma(A_i)$ of A_i in the plane $z = i$ (the i-th* layer*), such that:*

- *for every $1 \leqslant i \leqslant h(T)$ and for every cluster μ at level i, the point $a_i(\mu)$ representing μ in $\Gamma(A_i)$ is inside δ^i_μ; and*
- *for every three distinct clusters μ, ν, and ρ such that μ and ν are at the same level i, edge $(a_i(\mu), a_i(\nu))$ in $\Gamma(A_i)$ does not intersect δ^i_ρ.*

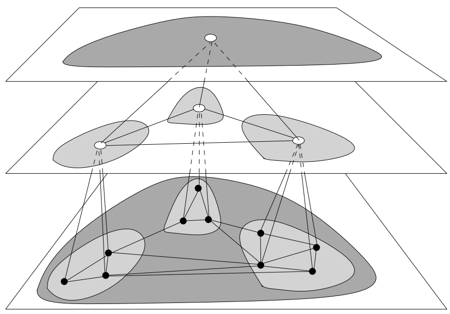


Figure 2: A multilayer drawing of a clustered graph.

Given a multilayer drawing of a clustered graph $C(G, T)$, tree T can be nicely visualized as follows. Map each internal node of T corresponding to a cluster μ at level i to the point $a_i(\mu)$, map each leaf of T corresponding to a vertex v in G to the point in Γ representing v, and finally map each edge of T to a straight-line segment between its end-points. The fact that every path in T from the root to a leaf has the same length and the fact that Γ is a convex straight-line drawing are easily shown to imply that no two edges of T cross when visualized in this fashion. See Figure 2 for an example of a multilayer drawing of a clustered graph together with its inclusion tree.

Multilayer drawings allow to simultaneously represent clustered graphs at different levels of abstractions. Namely distinct layers of a multilayer drawing represent cluster adjacency graphs at distinct levels of the inclusion tree. The property that point $a_i(\mu)$ is inside region δ^i_μ aims at preserving the user's mental map while changing the level of abstraction of the visualization. Further, the property that edge $(a_i(\mu), a_i(\nu))$ does not intersect region δ^i_ρ extends the absence of edge-region crossings to the distinct layers of the visualization.

The main result of this paper is the following:

Theorem 1 *Every c-planar flat clustered graph admits a multilayer drawing.*

We will prove Theorem 1 by showing an inductive algorithm that constructs a multilayer drawing of any c-planar flat clustered graph for an arbitrary drawing of its outer face satisfying certain geometric constraints.

Remark 1. Multilayer drawings strongly resemble *multilevel drawings*, defined by Eades and Feng in [9]. The only and yet fundamental difference between multilayer drawings and multilevel drawings is in the geometric objects representing the vertices of graph A_i in $\Gamma(A_i)$: points in multilayer drawings and arbitrary convex shapes in multilevel drawings. As shown in [9], a multilevel drawing of a clustered graph C can be easily obtained by first computing a straight-line convex drawing of C in the plane $z = 0$, by then vertically translating each cluster μ at level i to the plane $z = i$, and by finally using vertical translations of the edges of G to represent the edges of graphs A_i. This technique does not work for constructing multilayer drawings (see also Remark 2). A major source of difficulty in the construction of multilayer drawings is that all the edges of a graph A_i that are incident to the same vertex μ have to be incident to the same point $a_i(\mu)$ in $\Gamma(A_i)$, while in a multilevel drawing each of them can be incident to any point on the boundary of the convex shape representing μ in $\Gamma(A_i)$.

Remark 2. Not every convex straight-line drawing of a clustered graph can be "extended" to a multilayer drawing. Figure 3 shows a convex straight-line drawing Γ of a flat clustered graph $C(G, T)$ with cluster adjacency graph A such that, in any straight-line drawing $\Gamma(A)$ of A on the plane $z = 1$ in which $a(\alpha)$ is inside region δ^1_α for every cluster α in T, edge $(a(\mu), a(\rho))$ or edge $(a(\mu), a(\tau))$ of A intersects δ^1_ν.

Remark 3. The assumption that each path in T from the root to a leaf has the same length is necessary for a multilayer drawing of a clustered graph $C(G, T)$ to directly provide a nice visualization of T. In fact, without that assumption, it would not hold that any edge of T connects two vertices on

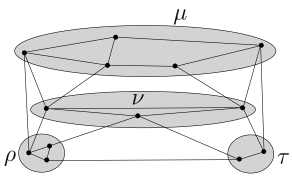


Figure 3: Illustration for Remark 2.

consecutive layers. However, if T is not such that each path from the root to a leaf has the same length, it suffices to introduce *dummy clusters* until this property is obtained. The effect of such a modification for the visualization of T is that each edge of T is represented by a sequence of straight-line segments, rather than by a single segment.

Remark 4. In the remainder of the paper, we will talk about multilayer drawings as two-dimensional drawings. That is, we will construct drawings of the cluster adjacency graph A of a flat clustered graph $C(G,T)$ in the plane $z=0$, rather than in the plane $z=1$. This allows us to more easily argue on whether the conditions that point $a(\mu)$ is inside δ^1_μ and that edge $(a(\mu),a(\nu))$ does not intersect δ^1_ρ are satisfied. Of course, given the two-dimensional drawing, the three-dimensional drawing can be easily obtained by vertically translating the drawing of A to the plane $z=1$.

Organization of the paper. In Section 2 we provide some definitions and state our main theorem, which implies Theorem 1; in Section 3 we provide some lemmata that will be used in the subsequent sections; in Section 4 we provide an outline of an inductive algorithm that proves our main theorem; in Sections 5 and 6 we prove the base cases and the inductive cases of such an algorithm, respectively; finally, in Section 7 we conclude and suggest an open problem.

2 The Main Theorem

In this section we state our main theorem, which directly implies Theorem 1.

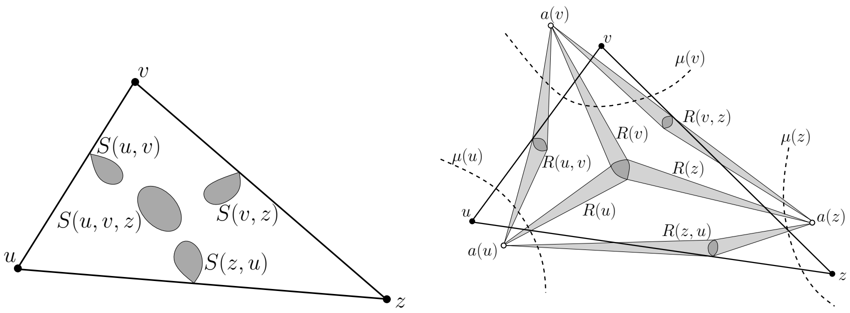
First, it suffices to restrict the attention to *maximal* c-planar flat clustered graphs, that is, to c-planar flat clustered graphs $C(G,T)$ such that G is a maximal planar graph. Indeed, if $C(G,T)$ is not maximal, then it can be augmented to a maximal clustered graph $C'(G',T)$ by adding dummy edges without loosing c-planarity [22]. Then a multilayer drawing of C' can be constructed and the inserted dummy edges can be deleted thus obtaining a multilayer drawing of C. In the following, all the considered clustered graphs are flat and maximal, even when not explicitly stated, and each clustered graph $C(G,T)$ is associated with a c-planar embedding that determines the faces of G.

We will denote a clustered graph also by $C(G,T,A)$, where A is the cluster adjacency graph of $C(G,T)$. The *outer face* of $C(G,T,A)$ is the clustered graph $C_o(G_o,T_o,A_o)$ such that G_o is the cycle delimiting the outer face of G, T_o is

T restricted to the clusters that contain vertices of G_o, and A_o is the cluster adjacency graph of $C_o(G_o, T_o)$.

We introduce some geometric concepts. We denote by $\sigma(p, R)$ the convex hull of a convex region R and a point p. We denote by $R_1 \cup R_2$ the union of two regions R_1 and R_2. For any two points p and q in the plane, we denote by $h(p, q)$ the half-line starting at p and passing through q. In the remainder of the paper we will call "cluster" both a vertex μ of the inclusion tree T of a clustered graph C and the simple closed region of the plane representing μ in a drawing of C.

Given a triangle (u, v, z), we define a *side region* $S(u, v)$ as a convex region that intersects segment $\overline{uv}$ in exactly one point and whose every other point is internal to (u, v, z); also, we define a *central region* $S(u, v, z)$ as a convex region entirely internal to (u, v, z). See Figure 4(a). Side and central regions are used to define *extension regions*. In the inductive algorithm that we will present in the next sections, extension regions are associated to a multilayer drawing Γ' of a subgraph $C'(G', T', A')$ of the clustered graph $C(G, T, A)$ to be drawn. The algorithm will draw the edges of A not in A' inside the extension regions associated with Γ'; the geometric properties of the extension regions detailed in Definition 2 guarantee that the edges of A not in A' do not cross edges of A' or clusters of T'. Hence, on one side such geometric properties have to be strong enough to guarantee the drawability of A, while on the other side they have to be weak enough to ensure that they hold inductively.



(a) (b)

Figure 4: (a) Side regions $S(u, v)$, $S(v, z)$, and $S(z, u)$, and central region $S(u, v, z)$. (b) Extension regions for a face $f = (u, v, z)$ if $\mu(u)$, $\mu(v)$, and $\mu(z)$ are distinct. The darker regions inside $R(u, v)$, $R(v, z)$, $R(z, u)$, and $R(u) \cup R(v) \cup R(z)$ are $S(u, v)$, $S(v, z)$, $S(z, u)$, and $S(u, v, z)$, respectively.

Definition 2 *Let* Γ *be a multilayer drawing of a flat clustered graph* $C(G, T, A)$*. Let* $f = (u, v, z)$ *be a face of* G*, where vertices* u*,* v*, and* z *appear in this clockwise order around* f*. The* extension regions for f *are defined as follows (see Figure 4(b)).*

- *If $\mu(u)$, $\mu(v)$, and $\mu(z)$ are all distinct, then let $S(u,v,z)$ be a central region and $S(u,v)$, $S(v,z)$, and $S(z,u)$ be side regions inside (u,v,z) such that:*
  - *$\sigma(u, S(u,v,z))$, $\sigma(u, S(u,v))$, and $\sigma(u, S(z,u))$ do not intersect each other and do not intersect any cluster except for $\mu(u)$;*
  - *$\sigma(v, S(u,v,z))$, $\sigma(v, S(u,v))$, and $\sigma(v, S(v,z))$ do not intersect each other and do not intersect any cluster except for $\mu(v)$;*
  - *$\sigma(z, S(u,v,z))$, $\sigma(z, S(v,z))$, and $\sigma(z, S(z,u))$ do not intersect each other and do not intersect any cluster except for $\mu(z)$;*
  - *for every point $p \in S(u,v,z)$, segments $\overline{pa(u)}$, $\overline{pa(v)}$, and $\overline{pa(z)}$ are in this clockwise order around p.*

  Then the extension regions for f are:

  1. *$R(u,v) = \sigma(a(u), S(u,v)) \cup \sigma(a(v), S(u,v))$;*
  2. *$R(v,z) = \sigma(a(v), S(v,z)) \cup \sigma(a(z), S(v,z))$;*
  3. *$R(z,u) = \sigma(a(z), S(z,u)) \cup \sigma(a(u), S(z,u))$;*
  4. *$R(u) = \sigma(a(u), S(u,v,z))$;*
  5. *$R(v) = \sigma(a(v), S(u,v,z))$; and*
  6. *$R(z) = \sigma(a(z), S(u,v,z))$.*

- *If $\mu(u) = \mu(v) \neq \mu(z)$, then let $S(v,z)$ and $S(z,u)$ be side regions inside (u,v,z) such that:*
  - *$\sigma(v, S(v,z))$ and $\sigma(u, S(z,u))$ do not intersect each other and do not intersect any cluster except for $\mu(u)$;*
  - *$\sigma(z, S(v,z))$ and $\sigma(z, S(z,u))$ do not intersect each other and do not intersect any cluster except for $\mu(z)$.*

  Then the extension regions for f are:

  1. *$R(v,z) = \sigma(a(u), S(v,z)) \cup \sigma(a(z), S(v,z))$; and*
  2. *$R(z,u) = \sigma(a(u), S(z,u)) \cup \sigma(a(z), S(z,u))$.*

- *If $\mu(u) = \mu(v) = \mu(z)$, then f has no extension regions.*

Next, we define *extensible drawings*, which are used throughout the remainder of the paper.

Definition 3 *A multilayer drawing of a flat clustered graph $C(G,T,A)$ is called* extensible *if, for each face f of G, extension regions for f can be drawn in such a way that:*

1. *for each face f of G, no extension region for f intersects an edge of A, except on its border;*

2. *for each face $f = (u, v, z)$ of G, no two distinct extension regions for f intersect, except on their borders, unless they both comprise a central region $S(u, v, z)$;*

3. *for each two faces f_1 and f_2 of G, where $f_1 \neq f_2$, no extension region for f_1 intersects an extension region for f_2, except on its border;*

4. *each extension region $R(u, v)$ for a face $f = (u, v, z)$ of G does not intersect any cluster other than $\mu(u)$ and $\mu(v)$; and*

5. *each extension region $R(u)$ for a face $f = (u, v, z)$ of G does not intersect any cluster other than $\mu(u)$.*

Figure 5 shows an extensible drawing of a flat clustered graph.

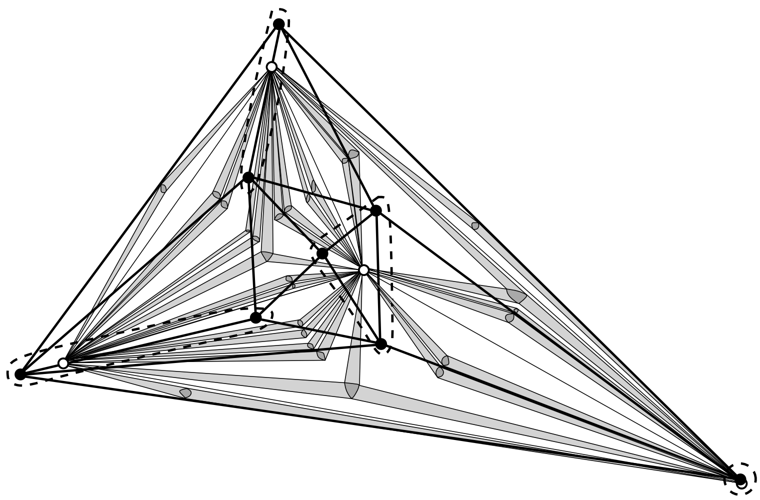


Figure 5: An extensible drawing of a flat clustered graph $C(G, T, A)$. Vertices of G are black dots and edges of G are thick lines. The borders of the clusters are dashed lines. Vertices of A are white disks and edges of A are thin lines. The extension regions are gray.

The inductive hypothesis of the algorithm, presented in the next section, is that an extensible drawing of a clustered graph can be constructed for an arbitrary extensible drawing of the outer face. Thus, we define the concept of *completing a drawing of the outer face.*

Definition 4 *An extensible drawing Γ of a flat clustered graph $C(G, T, A)$* completes *an extensible drawing Γ_o of the outer face $C_o(G_o, T_o, A_o)$ of C if Γ coincides with Γ_o when restricted to the vertices and edges of G_o, to the clusters in T_o, and to the vertices and edges of A_o, if all the vertices and edges of A/A_o lie in the extension regions of Γ_o, and if all the extension regions of Γ lie in the extension regions of Γ_o.*

We are now ready to state our main theorem.

Theorem 2 *Let C be a flat clustered graph. Then, for every extensible drawing Γ_o of the outer face C_o of C, there exists an extensible drawing Γ of C that completes Γ_o.*

Theorem 2 implies Theorem 1, since an extensible drawing is a multilayer drawing. In the following sections, we will prove a statement which is even stronger than the one in Theorem 2, namely that for every extensible drawing Γ_o of C_o, there exists an extensible drawing Γ of C that completes Γ_o, even if each cluster has to be represented by an arbitrary convex shape and if each vertex μ of A has to coincide with one of the vertices of G in μ. However, both such conditions are not necessary for our inductive proof to work, hence we omitted them from the statement of Theorem 2 and we invite the reader to observe how the drawings we construct actually satisfy such conditions.

3 Geometric and Topological Tools

In this section we present some lemmata that will be useful for the proofs in the remainder of the paper.

Consider a clustered graph $C(G, T)$ and, for any cluster $\mu \in T$, denote by $G[\mu]$ the subgraph of G induced by μ, that is, the subgraph of G whose vertices are those in μ and whose edges are those connecting two vertices in μ. We say that $C(G, T)$ is *c-connected* if, for every cluster $\mu \in T$, graph $G[\mu]$ is connected. We will make use of the following lemma.

Lemma 1 *Any maximal c-planar clustered graph is c-connected.*

Proof: Consider any maximal c-planar clustered graph $C(G, T)$ and suppose, for a contradiction, that it is not c-connected. Then there exists a cluster $\mu \in T$ such that $G[\mu]$ is not connected. Since G is a maximal planar graph, there exists a cycle $\mathcal{C}$ in G such that: (i) no vertex of $\mathcal{C}$ belongs to μ, and (ii) in any planar drawing Γ of G, $\mathcal{C}$ contains a vertex of μ in its interior and a vertex of μ in its exterior. It follows that, in any drawing Γ_C of C with no edge crossing and such that the closed simple region μ contains all and only the vertices that belong to μ, the boundary of μ intersects an edge of $\mathcal{C}$ at least twice, thus Γ_C is not c-planar. Hence, C is not c-planar, a contradiction which proves the lemma. □

In Section 6, we will manipulate a maximal c-planar flat clustered graph $C(G, T)$ in two different ways.

The operation *split along a separating triangle* takes as an input a maximal c-planar flat clustered graph $C(G, T)$ and a separating triangle (u', v', z') in G and returns two clustered graphs $C^1(G^1, T^1)$ and $C^2(G^2, T^2)$ defined as follows. Graph G^1 (G^2) is the subgraph of G induced by all the vertices external to (u', v', z') (resp. internal to (u', v', z')), by u', by v', and by z'; T^1 (T^2) is the subtree of T whose clusters contain at least one vertex of G^1 (resp. of G^2). See Figure 6. We have the following:

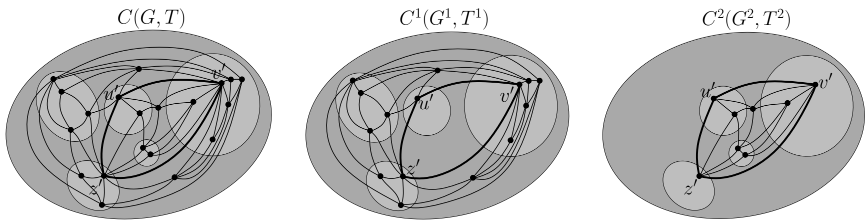


Figure 6: Illustration for the operation "split along a separating triangle".

Lemma 2 *The following statements hold:*

- $C^1(G^1, T^1)$ *and* $C^2(G^2, T^2)$ *are maximal c-planar flat clustered graphs;*
- *the number of vertices of each of* C^1 *and* C^2 *is strictly less than the number of vertices of* C*;*
- C_o *and* C_o^1 *are the same clustered graph; and*
- *denoting by* $C_o'(G_o', T_o')$ *the clustered graph such that* G_o' *is cycle* (u', v', z') *and* T_o' *is the subtree of* T *whose clusters contain at least one vertex of* G_o'*, we have that* C_o' *and* C_o^2 *are the same clustered graph.*

Proof: Since T^1 and T^2 are subtrees of T, it follows that $C^1(G^1, T^1)$ and $C^2(G^2, T^2)$ are flat clustered graphs. Since all the faces of G^1 (G^2), except for the one delimited by cycle (u', v', z'), are also faces of G, they are all delimited by cycles with three incident vertices. It follows that $C^1(G^1, T^1)$ and $C^2(G^2, T^2)$ are maximal flat clustered graphs. Finally, a c-planar drawing of $C^1(G^1, T^1)$ ($C^2(G^2, T^2)$) can be obtained from any c-planar drawing of $C(G, T)$ by removing vertices, edges, and clusters not in $C^1(G^1, T^1)$ (resp. not in $C^2(G^2, T^2)$). Thus, $C^1(G^1, T^1)$ and $C^2(G^2, T^2)$ are maximal c-planar flat clustered graphs.

Since (u', v', z') is a separating triangle, the number of vertices of each of C^1 and C^2 is strictly less than the number of vertices of C.

Since the cycle delimiting the outer face of G^1 is the same cycle delimiting the outer face of G, and since T^1 contains all the clusters that contain vertices of G^1, we have that C_o and C_o^1 are the same clustered graph.

Finally, since (u', v', z') delimits both the outer face of G_o' and the outer face of G^2, and since T_o' and T^2 contain all the clusters that contain u', v', or z', we have that C_o' and C_o^2 are the same clustered graph. □

The operation *contraction of an internal edge* takes as an input a maximal c-planar flat clustered graph $C(G, T)$ with no separating triangle and an internal edge (u', v') such that $\mu(u') = \mu(v')$, and returns a clustered graph $C'(G', T')$ defined as follows. Since G is maximal, since (u', v') is an internal edge, and since G contains no separating triangle, it follows that: (i) u' and v' have exactly two common neighbors, say z_1 and z_2, and (ii) vertices u', z_1, v', and z_2 appear in this clockwise order along cycle (u', z_1, v', z_2), with edge (u', v') in the interior of such a cycle. Denote by $u_0 = z_1, u_1 = v', u_2 = z_2, u_3, \ldots, u_l$ and by $v_0 = z_2, v_1 = u', v_2 = z_1, v_3, \ldots, v_m$ the clockwise orders of the neighbors of u' and of v' in G, respectively. Contract edge (u', v') to a vertex w, that is, delete u' and v' and their incident edges and insert a vertex w inside the face of $G'' = G - \{u, v\}$ where u and v used to lie. The clockwise order of the neighbors of w is $(z_1, v_3, \ldots, v_m, z_2, u_3, \ldots, u_l)$. Denote by G' the resulting graph. Also, denote by T' the tree obtained from T be removing vertices u' and v' and by inserting w as a child of $\mu(u')$. See Figure 7. We have the following.

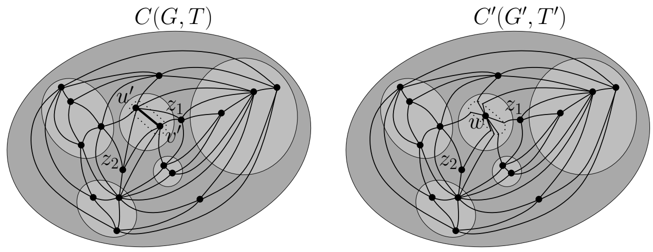


Figure 7: Illustration for the operation “contraction of an internal edge”.

Lemma 3 *The following statements hold:*

- $C'(G', T')$ *is a maximal c-planar flat clustered graph;*
- *the number of vertices of* C' *is strictly less than the number of vertices of* C*; and*
- C_o *and* C'_o *are the same clustered graph.*

Proof: By construction, T' has the same clusters as T, hence $C'(G', T')$ is a flat clustered graph. All the faces of G', except for the one in which u' and v' lie in G, are delimited by cycles with three incident vertices, because they are also faces of G. Further, the cycle delimiting the face of G' in which u' and v' lie in G is simple, as otherwise G would contain a separating triangle. It follows that G' is simple and maximal, hence $C'(G', T')$ is a maximal flat clustered graph. Finally, a c-planar drawing Γ' of $C'(G', T')$ can be obtained from any c-planar drawing Γ of $C(G, T)$ by drawing a simple closed region D in Γ slightly surrounding edge (u', v'), by deleting edges (u', v'), (u', z_1), and (u', z_2) from Γ, by deleting the part of each edge incident to u' or to v' in the interior of D, and by connecting all the points on the boundary of D and belonging to some edge to a point (representing w) in the interior of D. Thus, $C'(G', T')$ is a maximal c-planar flat clustered graph.

By construction, C' has exactly one less vertex than C.

Finally, since (u', v') is an internal edge of G, the cycle delimiting the outer face of G' coincides with the cycle delimiting the outer face of G. □

We now present a geometric lemma that will turn out to be more than useful in the upcoming proofs. Consider a maximal c-planar flat clustered graph $C(G, T, A)$ and its outer face $C_o(G_o, T_o, A_o)$. Denote by u, v, and z the clockwise order of the vertices along cycle G_o. Assume that $\mu(u) \neq \mu(v)$, $\mu(u) \neq \mu(z)$, and $\mu(v) \neq \mu(z)$. Consider an extensible drawing Γ_o of C_o. Let p be any point in $S(u, v, z)$. Consider the wedge W_u with an angle smaller than 180° and delimited by $h(p, u)$ and $h(p, a(u))$. We have the following.

Lemma 4 *Neither v, nor z, nor $a(v)$, nor $a(z)$ is contained in W_u.*

Proof: Suppose that the clockwise rotation around p bringing $h(p, u)$ to coincide with $h(p, a(u))$ is smaller than 180°. The case in which the counter-clockwise rotation around p bringing $h(p, u)$ to coincide with $h(p, a(u))$ is smaller than 180° can be discussed analogously.

Suppose, for a contradiction, that z is in W_u. Since the clockwise order of the vertices around the outer face of G is u, v, and z, it follows that v lies to the left of $h(u, z)$; however, this implies that triangle (u, v, z) does not contain p in its interior, a contradiction.

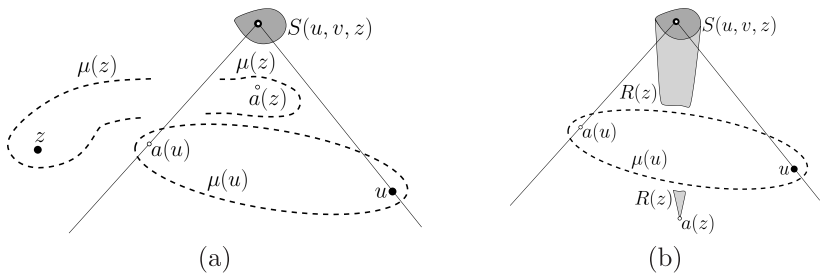


Figure 8: Illustration for the proof of Lemma 4.

If $a(z)$ belongs to W_u, then we distinguish two cases. If $a(z)$ is in the bounded region B of the plane delimited by segment $\overline{pa(u)}$, by segment $\overline{pu}$, and by the border of cluster $\mu(u)$, as in Figure 8(a), then $\mu(z)$ intersects either cluster $\mu(u)$, or region $\sigma(u, S(u, v, z))$, or extension region $R(u)$, given that z is not in W_u, thus contradicting the assumption that Γ_o is an extensible drawing. Otherwise, $a(z)$ is not in B, as in Figure 8(b); however, this implies that $R(z)$ intersects cluster $\mu(u)$, thus contradicting the assumption that Γ_o is an extensible drawing. It follows that $a(z)$ does not belong to W_u.

If $a(v)$ is in W_u, then, since segments $\overline{pa(u)}$, $\overline{pa(v)}$, and $\overline{pa(z)}$ come in this clockwise order around p, vertex $a(z)$ is also in W_u, a contradiction.

Finally, if v is in W_u, we distinguish two cases. If v is in B, then $\mu(v)$ intersects either $\mu(u)$, or region $\sigma(u, S(u, v, z))$, or extension region $R(u)$, given that $a(v)$ is not in W_u; if v is not in B, then $\sigma(v, S(u, v, z))$ intersects cluster $\mu(u)$. In both cases, the assumption that Γ_o is an extensible drawing is contradicted, hence v does not belong to W_u. This concludes the proof of the lemma. □

4 Algorithm Outline

Our proof of Theorem 2 consists of an algorithm that, given a maximal c-planar flat clustered graph $C(G, T, A)$ and an extensible drawing Γ_o of the outer face $C_o(G_o, T_o, A_o)$ of $C(G, T, A)$, constructs an extensible drawing Γ of $C(G, T, A)$ that completes Γ_o. The algorithm works by induction on the number of vertices of G. In this section we present an outline of the algorithm.

The induction distinguishes five cases:

- (Inductive Case 1) G contains a separating triangle.
- (Inductive Case 2) G contains no separating triangle and contains an internal edge (u', v') such that $\mu(u') = \mu(v')$.
- (Base Case 1) G has no internal vertices.
- (Base Case 2) G is K_4 and it does not contain any internal edge (x, y) with $\mu(x) = \mu(y)$.
- (Base Case 3) G contains more than one internal vertex, does not contain any separating triangle, and does not contain any internal edge (x, y) with $\mu(x) = \mu(y)$.

The inductive cases of the algorithm strongly resemble the ones in the famous algorithm by Fary [12], proving that every plane graph admits a straight-line planar drawing.

In Inductive Case 1, we consider a separating triangle (u', v', z'). We apply the operation "split along a separating triangle", defined in Section 3, in order to split $C(G, T)$ in two smaller maximal c-planar flat clustered graphs $C^1(G^1, T^1)$ and $C^2(G^2, T^2)$ (see also Lemma 2). We inductively compute an extensible drawing Γ^1 of C^1 completing Γ_o; moreover, we inductively compute an extensible drawing Γ^2 of C^2 completing Γ^2_o, where Γ^2_o is the drawing of C^2_o in Γ^1. Plugging Γ^2 into Γ^1 provides a drawing Γ of C.

In Inductive Case 2, we consider an internal edge (u', v') such that $\mu(u') = \mu(v')$. We apply the operation "contraction of an internal edge", defined in Section 3, in order to transform $C(G, T)$ into a smaller maximal c-planar flat clustered graph $C'(G', T')$ (see also Lemma 3). We inductively compute an extensible drawing Γ' of C' completing Γ_o; further, we suitably replace the vertex w edge (u', v') has been contracted to with a drawing of edge (u', v'), thus obtaining an extensible drawing Γ of C completing Γ_o.

If neither Inductive Case 1 nor Inductive Case 2 applies, then we are in a base case, that is, we can provide an algorithm that constructs an extensible drawing Γ of C completing Γ_o without using induction. This is trivial in Base Case 1, in which $C(G, T)$ coincides with $C_o(G_o, T_o)$ (and hence Γ coincides with Γ_o). However, the task is not trivial in Base Cases 2 and 3. In these cases, involved geometric considerations lead to determine a drawing for the vertices of G not in G_o, the clusters of T not in T_o, and the vertices of A not in A_o, thus obtaining an extensible drawing Γ of C completing Γ_o.

We conclude this section by proving that the case distinction we presented is complete.

Lemma 5 *Let $C(G,T)$ be a maximal c-planar flat clustered graph. Then exactly one among Inductive Cases 1-2 and Base Cases 1-3 applies to $C(G,T)$.*

Proof: Suppose that G has a separating triangle. Then Inductive Case 1 applies to $C(G,T)$. By definition, Inductive Case 2 and Base Case 3 do not apply to $C(G,T)$. In order for G to contain a separating triangle, the number of vertices of G is at least five. Hence, Base Cases 2-3 do not apply to $C(G,T)$.

Assume next that G has no separating triangle (hence Inductive Case 1 does not apply to $C(G,T)$). If G has an internal edge (u',v') such that $\mu(u')=\mu(v')$, then Inductive Case 2 applies to $C(G,T)$. By definition, Base Cases 2-3 do not apply to $C(G,T)$. Since G has an internal edge, it also has an internal vertex, hence Base Case 1 does not apply to $C(G,T)$.

Assume next that G has no separating triangle and no internal edge (u',v') such that $\mu(u')=\mu(v')$ (hence Inductive Cases 1 and 2 do not apply to $C(G,T)$). If G has no internal vertex, then Base Case 1 applies to $C(G,T)$ and, by definition, Base Cases 2 and 3 do not apply to $C(G,T)$. If G has exactly one internal vertex, then Base Case 2 applies to $C(G,T)$ and, by definition, Base Cases 1 and 3 do not apply to $C(G,T)$. If G has more than one internal vertex, then Base Case 3 applies to $C(G,T)$ and, by definition, Base Cases 1 and 2 do not apply to $C(G,T)$. □

5 Base Cases

In this section we present the three base cases of the induction. Denote by (u,v,z) the clockwise order of the vertices along cycle G_o.

Base Case 1: *G has no internal vertices.* In this case, $C=C_o$ and the extensible drawing Γ_o of C_o is an extensible drawing Γ of C that completes Γ_o.

Base Case 2: *G is K_4 and it does not contain any internal edge (x,y) with $\mu(x)=\mu(y)$.* Refer to Figure 9. Let x and f be the unique internal vertex and internal face of G, respectively. Select any point p inside $S(u,v,z)$ (if $\mu(u)$, $\mu(v)$, and $\mu(z)$ are all different), or inside $S(v,z)$ (if $\mu(u)=\mu(v)\neq\mu(z)$). The cases in which $\mu(u)=\mu(z)\neq\mu(v)$ or $\mu(v)=\mu(z)\neq\mu(u)$ can be treated analogously to the case in which $\mu(u)=\mu(v)\neq\mu(z)$. Observe that $\mu(u)=\mu(v)=\mu(z)$ does not hold, given that C is c-planar and that $\mu(x)\neq\mu(u)$. Draw both x and $a(x)$ at p. Draw the edges of G and A not in C_o as straight-line segments.

We next draw extension regions for the three internal faces $f_1=(u,v,x)$, $f_2=(v,z,x)$, and $f_3=(z,u,x)$ of G. We distinguish two cases, depending on whether $\mu(u)$, $\mu(v)$, and $\mu(z)$ are all different, or $\mu(u)=\mu(v)\neq\mu(z)$.

First, suppose that $\mu(u)$, $\mu(v)$, and $\mu(z)$ are all different.

- Extension regions $R(u,v)$ for f_1, $R(v,z)$ for f_2, and $R(z,u)$ for f_3 coincide with extension regions $R(u,v)$, $R(v,z)$, and $R(z,u)$ for f, respectively.

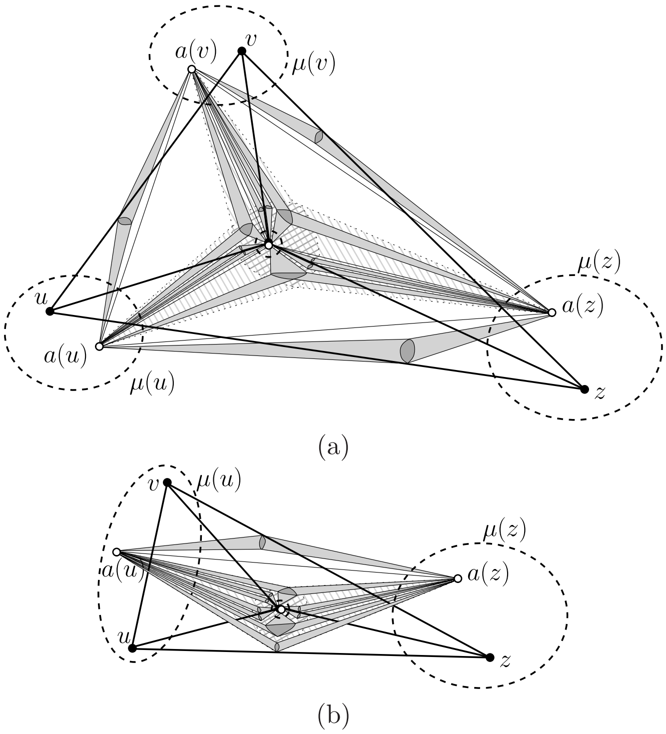


Figure 9: Base Case 2, if (a) $\mu(u)$, $\mu(v)$, and $\mu(z)$ are all different, or if (b) $\mu(u) = \mu(v) \neq \mu(z)$.

- Side regions $S(v, x)$ for f_1 and $S(v, x)$ for f_2 are drawn as small regions inside $S(u, v, z)$ touching (v, x) in two points p and q very close to x, such that q is farther from x than p or closer to x than p, depending on whether $a(v)$ is to the left or to the right of $h(x, v)$, respectively. This placement implies that extension regions $R(v, x)$ for f_1 and $R(v, x)$ for f_2 do not intersect, except on their borders.

- Extension regions $R(x, u)$ for f_1, $R(x, u)$ for f_3, $R(z, x)$ for f_2, and $R(z, x)$ for f_3 are drawn analogously.

- In order to define $S(u, v, x)$, we first select a point p with the following properties: (i) p is in the interior of $S(u, v, z)$; (ii) p is inside f_1; (iii) p is to the right of $h(a(v), x)$; (iv) p is to the left of $h(a(u), x)$; and (v) segments $\overline{pa(u)}$, $\overline{pa(v)}$, and $\overline{pa(x)}$ do not intersect the extension regions $R(v, x)$ and $R(x, u)$ for f_1. Observe a point p satisfying properties (i)–(iv) exists by Lemma 4 and since Γ_o is an extensible drawing. Further, p satisfies property (v) provided that side regions $S(v, x)$ and $S(x, u)$ for f_1 are small enough and sufficiently close to x. Figure 10 illustrates the choice of p for different arrangements of points u, v, x, $a(u)$, and $a(v)$. Then $S(u, v, x)$ is any small convex region surrounding p. An immediate consequence of such a construction is that extension regions $R(u)$, $R(v)$, and $R(x)$ do not intersect extension regions $R(v, x)$ and $R(x, u)$ for f_1.

- Extension regions $R(v)$, $R(z)$, and $R(x)$ for f_2 and $R(z)$, $R(u)$, and $R(x)$ for f_3 are constructed analogously.

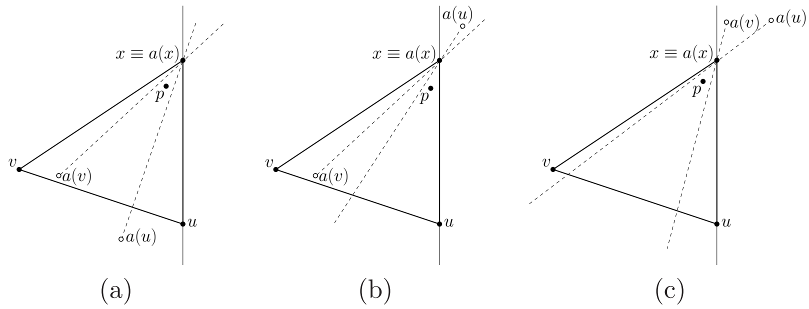


Figure 10: Choice of the point p around which region $S(u, v, x)$ is constructed.

Second, suppose that $\mu(u) = \mu(v) \neq \mu(z)$. Extension region $R(v, z)$ for f_2 coincides with extension region $R(v, z)$ for f. The construction of all other extension regions is the same as in the case in which $\mu(u)$, $\mu(v)$, and $\mu(z)$ are all different, with the only difference that the side and central regions that define the extension regions for f_1, f_2, and f_3 all lie inside the side region $S(z, u)$ that defines the extension region $R(z, u)$ for f (rather than inside the central region $S(u, v, z)$, which in this case does not exist).

It remains to draw cluster $\mu(x)$. Such a cluster is drawn as an arbitrary small region surrounding x. Denote by Γ the resulting drawing.

Lemma 6 *Γ is an extensible drawing of C completing Γ_o.*

Proof: We prove the statement in the case in which $\mu(u)$, $\mu(v)$, and $\mu(z)$ are all different. The proof for the cases in which two of such clusters coincide is analogous. We will prove the lemma by proving that: (i) Γ is a c-planar drawing; (ii) Γ is a convex drawing; (iii) Γ is a multilayer drawing; (iv) the extension regions satisfy the properties of Definition 2; and (v) Γ is an extensible drawing. Figure 11 describes the structure of the proof of the lemma.

C-planar drawing: Vertex x lies inside cycle (u, v, z), hence Γ has no edge crossings. Cluster $\mu(x)$ lies inside central region $S(u, v, z)$ which, by definition of extensible drawing, has no intersection with clusters $\mu(u)$, $\mu(v)$, and $\mu(z)$. Hence, Γ has no region-region crossings. Since $\mu(x)$ is drawn as an arbitrary small region inside $S(u, v, z)$, which lies inside cycle (u, v, z), cluster $\mu(x)$ does not intersect any edge of G other than (u, x), (v, x), and (z, x). Since $\sigma(u, S(u, v, z))$ has no intersection with any cluster other than $\mu(u)$, then edge

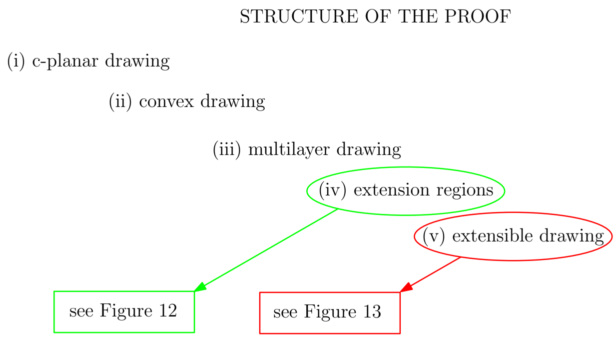


Figure 11: Structure of the proof of Lemma 6.

(u, x), which lies inside $\sigma(u, S(u, v, z))$, has no intersections with clusters $\mu(v)$ and $\mu(z)$. Analogously, edge (v, x) has no intersections with clusters $\mu(u)$ and $\mu(z)$, and edge (z, x) has no intersections with clusters $\mu(u)$ and $\mu(v)$. Hence, Γ has no edge-region crossings. It follows that Γ is a c-planar drawing.

Convex drawing: Clusters $\mu(u)$, $\mu(v)$, and $\mu(z)$ are convex, since Γ_o is an extensible drawing, and cluster $\mu(x)$ is convex by construction. It follows that Γ is a straight-line convex drawing.

Multilayer drawing: Vertices $a(u)$, $a(v)$, and $a(z)$ are inside $\mu(u)$, $\mu(v)$, and $\mu(z)$, respectively, since Γ_o is an extensible drawing, and vertex $a(x)$ is inside $\mu(x)$, by construction. Since Γ_o is an extensible drawing, $R(u)$, $R(v)$, and $R(z)$ do not intersect any of edges $(a(u), a(v))$, $(a(v), a(z))$, and $(a(z), a(u))$, except on their borders. Since $a(x)$ is inside $S(u, v, z)$, then edges $(a(u), a(x))$, $(a(v), a(x))$, and $(a(z), a(x))$ lie inside $R(u)$, $R(v)$, and $R(z)$, respectively. This implies that the drawing of A is planar, that edge $(a(u), a(x))$ has no intersection with clusters $\mu(v)$ and $\mu(z)$, that edge $(a(v), a(x))$ has no intersection with clusters $\mu(u)$ and $\mu(z)$, and that edge $(a(z), a(x))$ has no intersection with clusters $\mu(u)$ and $\mu(v)$. Moreover, none of edges $(a(u), a(v))$, $(a(v), a(z))$, and $(a(z), a(u))$ has intersection with cluster $\mu(x)$, since such a cluster is inside $S(u, v, z)$ and since Γ_o is an extensible drawing. It follows that Γ is a multilayer drawing.

Extension regions: We first prove that central region $S(u, v, x)$ and side regions $S(u, v)$, $S(v, x)$, and $S(x, u)$ have the properties required in Definition 2. Refer to Figure 12.

By construction, central region $S(u, v, x)$ and side regions $S(u, v)$, $S(v, x)$, and $S(x, u)$ lie inside f_1, central region $S(v, z, x)$ and side regions $S(v, z)$, $S(z, x)$, and $S(x, v)$ lie inside f_2, and central region $S(z, u, x)$ and side regions $S(z, u)$, $S(u, x)$, and $S(x, z)$ lie inside f_3.

Since Γ_o is an extensible drawing, region $\sigma(u, S(u, v)) \cup \sigma(v, S(u, v))$ does not intersect region $\sigma(u, S(u, v, z))$, except on its border, hence it does not intersect regions $\sigma(u, S(u, v, x))$ and $\sigma(u, S(u, x))$, except on its border, as such regions entirely lie inside $\sigma(u, S(u, v, z))$. Analogously, region $\sigma(u, S(u, v)) \cup \sigma(v, S(u, v))$ does not intersect regions $\sigma(v, S(u, v, x))$ and $\sigma(v, S(v, x))$, except

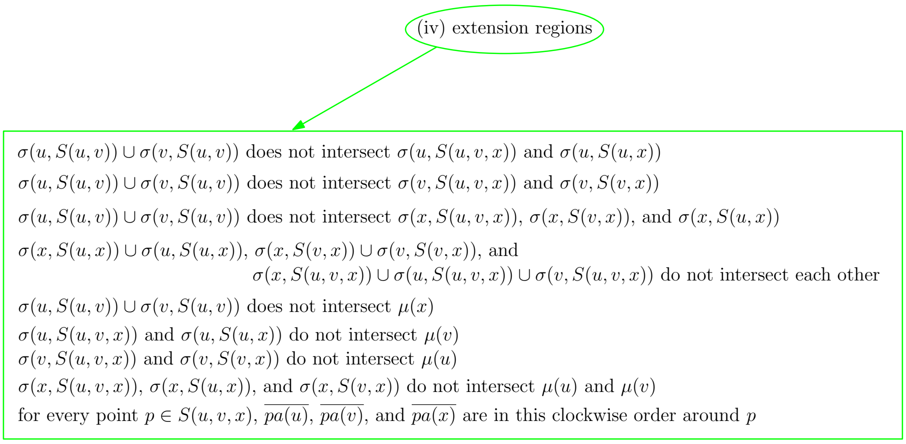


Figure 12: Structure of the proof that central region $S(u,v,x)$ and side regions $S(u,v)$, $S(v,x)$, and $S(x,u)$ have the properties required in Definition 2.

on its border. Moreover, region $\sigma(u, S(u,v)) \cup \sigma(v, S(u,v))$ does not intersect regions $\sigma(x, S(u,v,x))$, $\sigma(x, S(v,x))$, and $\sigma(x, S(u,x))$, as such regions entirely lie inside $S(u,v,z)$. Regions $\sigma(x, S(u,x)) \cup \sigma(u, S(u,x))$, $\sigma(x, S(v,x)) \cup \sigma(v, S(v,x))$, and $\sigma(x, S(u,v,x)) \cup \sigma(u, S(u,v,x)) \cup \sigma(v, S(u,v,x))$ do not intersect each other, since by construction the first two regions are arbitrarily close to segments $\overline{ux}$ and $\overline{vx}$, respectively, and as the third region is arbitrarily close to $\overline{pu} \cup \overline{pv} \cup \overline{px}$, with p being a point inside triangle (u,v,x) such that $S(u,v,x)$ is an arbitrarily small region surrounding p. Analogous considerations prove the disjointness of the regions inside f_2 and f_3.

Since Γ_o is an extensible drawing, region $\sigma(u, S(u,v)) \cup \sigma(v, S(u,v))$ does not intersect cluster $\mu(x)$, as such a cluster is inside $S(u,v,z)$. Moreover, regions $\sigma(u, S(u,v,x))$ and $\sigma(u, S(u,x))$ do not intersect cluster $\mu(v)$, as such regions lie inside $\sigma(u, S(u,v,z))$; analogously, $\sigma(v, S(u,v,x))$ and $\sigma(v, S(v,x))$ do not intersect cluster $\mu(u)$. Furthermore, regions $\sigma(x, S(u,v,x))$, $\sigma(x, S(u,x))$, and $\sigma(x, S(v,x))$ do not intersect clusters $\mu(u)$ and $\mu(v)$ as such regions lie inside $S(u,v,z)$. Analogous considerations prove the disjointness of the regions inside f_2 and f_3 with respect to the clusters containing vertices incident to such faces.

We prove that, for every point $q \in S(u,v,x)$, segments $\overline{qa(u)}$, $\overline{qa(v)}$, and $\overline{qa(x)}$ are in this clockwise order around q. By construction, $S(u,v,x)$ is entirely to the right of $h(a(v), a(x))$ and to the left of $h(a(u), a(x))$. Hence, the graph whose vertices are $a(u)$, q, $a(v)$, and $a(x)$ and whose edges are $(a(u), q)$, $(q, a(v))$, $(a(v), a(x))$, $(a(x), a(u))$, and $(a(x), q)$ is planar in Γ, and the clockwise order of the vertices along its outer face is $a(u)$, q, $a(v)$, and $a(x)$. The statement follows. Analogous considerations hold for the points in $S(v,z,x)$ and in $S(z,u,x)$.

Extensible drawing: Next, we show that the extension regions for f_1, f_2, and f_3 satisfy the five properties that are required for Γ to be an extensible drawing, as in Definition 3. Figure 13 illustrates the structure of the proof that Γ is an extensible drawing.

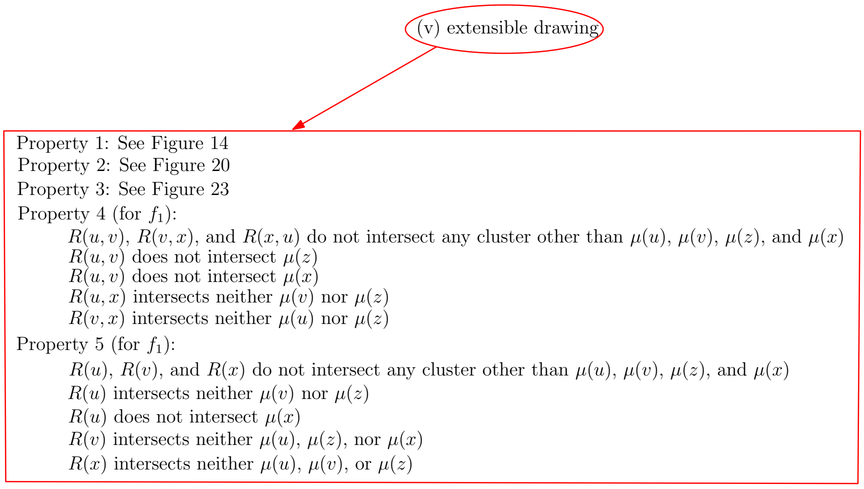


Figure 13: Structure of the proof that Γ satisfies Properties 1–5 of Definition 3.

Property 1. Figure 14 illustrates the structure of the proof that Γ satisfies Property 1 of Definition 3. We prove that no extension region for f_1 intersects an edge of A, the proofs for the extension regions for f_2 and f_3 being analogous. First, all the extension regions for f_1 are inside region $R(u,v) \cup R(v,z) \cup R(z,u) \cup R(u) \cup R(v) \cup R(z)$ (where $R(u,v)$, $R(v,z)$, $R(z,u)$, $R(u)$, $R(v)$, and $R(z)$ are extension regions for f), hence they intersect neither $(a(u),a(v))$, $(a(u),a(z))$, nor $(a(v),a(z))$, except on their borders, since Γ_o is an extensible drawing. Extension region $R(u,v)$ for f_1 does not intersect any edge of A incident to $a(x)$, given that any such an edge is inside region $R(u) \cup R(v) \cup R(z)$ (where $R(u)$, $R(v)$, and $R(z)$ are extension regions for f), given that extension region $R(u,v)$ for f_1 coincides with extension region $R(u,v)$ for f, and given that Γ_o is an extensible drawing.

It remains to prove that extension regions $R(u,x)$, $R(v,x)$, $R(u)$, $R(v)$, and $R(x)$ for f_1 do not intersect any edge of A incident to $a(x)$, except possibly on their borders, provided that regions $S(u,x)$, $S(v,x)$, and $S(u,v,x)$ are sufficiently small.

- We show that extension region $R(u,x)$ for f_1 does not intersect edge $(a(u),a(x))$, except on its border, provided that $S(u,x)$ is sufficiently small. We distinguish three cases. If $a(u)$ lies on the same side as v with respect to $h(u,x)$ (see Figure 15(a)), then it suffices to choose $S(u,x)$ sufficiently small so that it fits inside the wedge delimited by $h(x,u)$ and $h(x,a(u))$. If $a(u)$ lies on the opposite side of v with respect to $h(u,x)$ (see Figure 15(b)), then it suffices to choose $S(u,x)$ sufficiently small so

Property 1 (for f_1):

- no extension region for f_1 intersects any of $\{(a(u), a(v)), (a(u), a(z)), (a(v), a(z))\}$
- $R(u, v)$ intersects neither $(a(u), a(x))$,$(a(v), a(x))$, nor $(a(z), a(x))$
- $R(u, x)$ does not intersect $(a(u), a(x))$
  - if $a(u)$ and v are on the same side w.r.t. to $h(u, x)$
  - if $a(u)$ and v are not on the same side w.r.t. to $h(u, x)$
  - if $a(u)$, u, and x are collinear
- $R(v, x)$ does not intersect $(a(v), a(x))$
- $R(u, x)$ does not intersect $(a(v), a(x))$
- $R(v, x)$ intersects neither $(a(u), a(x))$ nor $(a(z), a(x))$, and $R(u, x)$ does not intersect $(a(z), a(x))$
- $R(u)$ does not intersect $(a(u), a(x))$
- $R(v)$ does not intersect $(a(v), a(x))$
- $R(u)$ does not intersect $(a(v), a(x))$
  - if $a(v)$ lies to the right of $h(v, x)$ and to the left of $h(u, x)$ and $a(u)$ is to the right of $h(a(v), a(x))$ or $a(u)$ is to the left of $h(a(v), a(x))$ } Case 1
  - if $a(v)$ lies to the left of $h(v, x)$ and to the left of $h(u, x)$ and $a(u)$ lies to the right of $h(v, x)$ or $a(u)$ lies to the left of $h(v, x)$ } Case 2
  - if $a(v)$ lies to the left of $h(v, x)$ and to the right of $h(u, x)$ and $a(u)$ is to the right of $h(v, x)$ or to the left of $h(u, x)$ or $a(u)$ lies to the left of $h(v, x)$ and to the right of $h(u, x)$ } Case 3
- $R(v)$ does not intersect $(a(u), a(x))$
- $R(u)$ does not intersect $(a(z), a(x))$
  - if $a(z)$ is to the right of $h(a(u), a(x))$
  - if $a(z)$ is to the left of $h(a(u), a(x))$ and $a(v)$ lies outside triangle $(a(x), a(u), r)$
  - if $a(z)$ is to the left of $h(a(u), a(x))$ and $a(v)$ lies inside triangle $(a(x), a(u), r)$
- $R(v)$ does not intersect $(a(z), a(x))$
- $R(x)$ intersects neither $(a(u), a(x))$,$(a(v), a(x))$, nor $(a(z), a(x))$

Figure 14: Structure of the proof that Γ satisfies Property 1 of Definition 3.

that it lies on the same side as u with respect to $h(a(u), x)$. Finally, if x, u, and $a(u)$ are collinear, then $R(u, x)$ does not intersect edge $(a(u), a(x))$, except on its border, for any choice of $S(u, x)$ inside (u, v, z).

- It can be proved analogously that extension region $R(v, x)$ for f_1 does not intersect edge $(a(v), a(x))$, except on its border, provided that $S(v, x)$ is sufficiently small.

- We prove that extension region $R(u, x)$ for f_1 does not intersect edge $(a(v), a(x))$, except on its border. For a contradiction, suppose it does. Then vertex $a(v)$ has to lie in the wedge with an angle smaller than $180°$ delimited by $h(x, u)$ and $h(x, a(u))$, as otherwise it would suffice to pick $S(u, x)$ as a sufficiently small region in order to avoid intersections between $R(u, x)$ and $(a(v), a(x))$ (see Figure 15(c)). However, by Lemma 4, vertex $a(v)$ does not lie in the wedge delimited by $h(x, u)$ and $h(x, a(u))$.

- It can be analogously proved that extension region $R(v, x)$ for f_1 does not intersect edge $(a(u), a(x))$ or edge $(a(z), a(x))$, except on its border, provided that $S(v, x)$ is sufficiently small, and that extension region $R(u, x)$ for f_1 does not intersect edge $(a(z), a(x))$, except on its border, provided that $S(u, x)$ is sufficiently small.

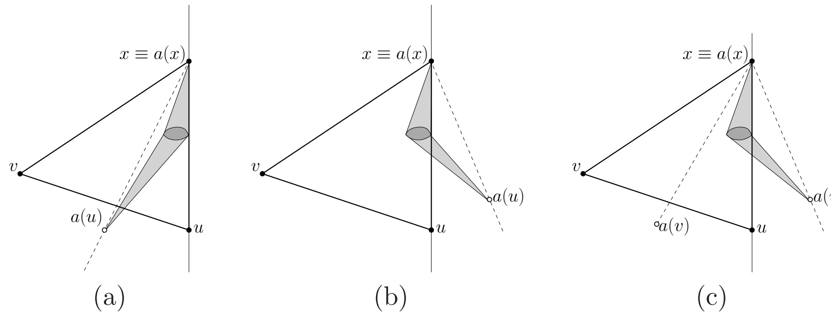


Figure 15: (a) Extension region $R(u, x)$ for f_1 does not intersect edge $(a(u), a(x))$ when $a(u)$ lies on the same side as v with respect to $h(u, x)$. (b) Extension region $R(u, x)$ for f_1 does not intersect edge $(a(u), a(x))$ when $a(u)$ lies on the opposite side of v with respect to $h(u, x)$. (c) Extension region $R(u, x)$ for f_1 does not intersect edge $(a(v), a(x))$ when $a(v)$ is not in the wedge with an angle smaller than 180° delimited by $h(x, u)$ and $h(x, a(u))$.

- Extension region $R(u)$ for f_1 does not intersect edge $(a(u), a(x))$, except on its border, provided that the point p around which region $S(u, v, x)$ is constructed is not on the line through $a(u)$ and $a(x)$ and provided that region $S(u, v, x)$ is small enough.

- It can be proved analogously that extension region $R(v)$ for f_1 does not intersect edge $(a(v), a(x))$, except on its border, provided that region $S(u, v, x)$ is small enough.

- Next, we prove that extension region $R(u)$ for f_1 does not intersect edge $(a(v), a(x))$. We distinguish three cases, based on the position of $a(v)$ with respect to triangle (u, v, x) (see Figure 16). In *Case 1*, vertex $a(v)$ lies to the left of $h(u, x)$ and to the right of $h(v, x)$. In *Case 2*, vertex $a(v)$ lies to the left of $h(v, x)$ and to the left of $h(u, x)$. In *Case 3*, vertex $a(v)$ lies to the right of $h(u, x)$ and to the left of $h(v, x)$. No other cases are possible. Namely, $a(v)$ does not lie to the right of $h(u, x)$ and to the right of $h(v, x)$, as otherwise Lemma 4 would be violated (with u being inside the wedge with an angle smaller than 180° delimited by $h(x, v)$ and $h(x, a(v))$.

  Case 1. By construction, region $S(u, v, x)$ is to the right of $h(a(v), a(x))$. Hence, if $a(u)$ is also to the right of $h(a(v), a(x))$, then $R(u)$ does not intersect edge $(a(v), a(x))$ (see Figure 17(a)). Suppose that $a(u)$ is to the left of $h(a(v), a(x))$. Then $a(u)$ does not lie to the left of $h(u, x)$, as otherwise the wedge with an angle smaller than 180° and delimited by $h(x, u)$ and $h(x, a(u))$ would contain $a(v)$, thus violating Lemma 4. Hence, $a(u)$ is to the right of $h(u, x)$. By construction, $S(u, v, x)$ is entirely to the left of $h(a(u), a(x))$, hence $R(u)$ and $(a(v), a(x))$ are separated by $h(a(u), a(x))$, and thus they do not intersect (see Figure 17(b)).

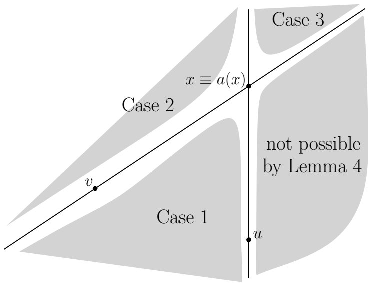


Figure 16: Possible locations of $a(v)$ with respect to triangle (u, v, x).

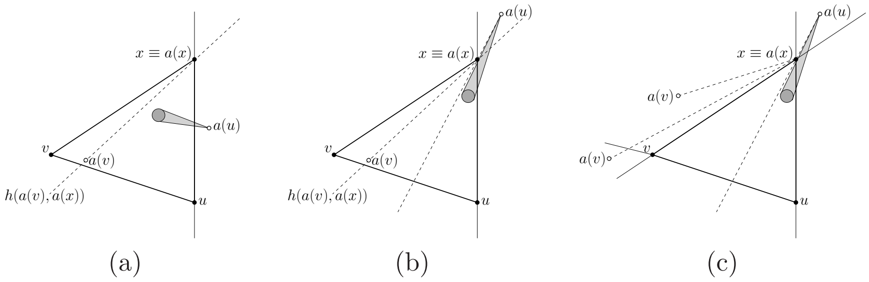
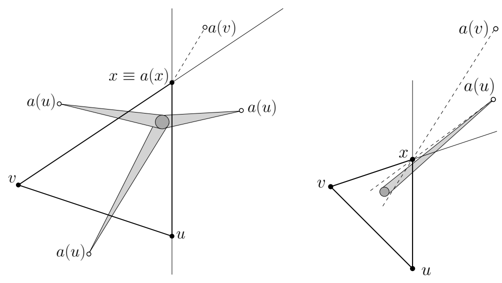


Figure 17: (a)–(b) Illustration for the proof that extension region $R(u)$ for f_1 does not intersect edge $(a(v), a(x))$ in Case 1. (c) Illustration for the proof that extension region $R(u)$ for f_1 does not intersect edge $(a(v), a(x))$ in Case 2.

Case 2. If $a(u)$ lies to the right of $h(v, x)$, then extension region $R(u)$ and edge $(a(v), a(x))$ are separated by $h(v, x)$, hence they do not intersect. Suppose that $a(u)$ is to the left of $h(v, x)$. Then $a(u)$ does not lie to the left of $h(u, x)$, as otherwise the wedge with an angle smaller than 180° and delimited by $h(x, u)$ and $h(x, a(u))$ would contain vertex v, thus violating Lemma 4. Hence, $a(u)$ is to the right of $h(u, x)$. By construction, $S(u, v, x)$ entirely lies to the left of $h(a(u), a(x))$, hence $R(u)$ and $(a(v), a(x))$ are separated by $h(a(u), a(x))$ and they do not intersect (see Figure 17(c)).

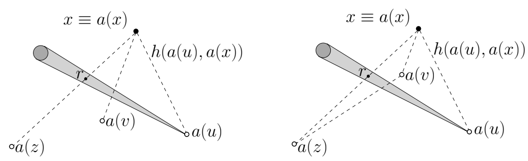
Case 3. By construction, region $S(u, v, x)$ is entirely inside f_1. Hence, if $a(u)$ is to the right of $h(v, x)$ or to the left of $h(u, x)$, then $R(u)$ and $(a(v), a(x))$ are separated by $h(v, x)$ or by $h(u, x)$, respectively, hence they do not intersect (see Figure 18(a)). Assume that $a(u)$ is to the left of $h(v, x)$ and to the right of $h(u, x)$. Then we have that $a(u)$ is to the left of $h(a(v), a(x))$, as otherwise Lemma 4 would be violated, with $a(v)$ being inside the wedge with an angle smaller than 180° and delimited by $h(x, u)$ and $h(x, a(u))$. By construction, $S(u, v, x)$ entirely lies to the left of $h(a(u), a(x))$, hence $R(u)$ and $(a(v), a(x))$ are separated by $h(a(u), a(x))$ and they do not intersect (see Figure 18(b)).



(a) (b)

Figure 18: Illustration for the proof that extension region $R(u)$ for f_1 does not intersect edge $(a(v), a(x))$ in Case 3. (a) $a(u)$ is to the right of $h(v, x)$ or to the left of $h(u, x)$. (b) $a(u)$ is to the left of $h(v, x)$ and to the right of $h(u, x)$.

- It can be proved analogously that extension region $R(v)$ for f_1 does not intersect edge $(a(u), a(x))$.
- Next, we prove that extension region $R(u)$ for f_1 does not intersect edge $(a(z), a(x))$. By construction, extension region $R(u)$ for f_1 is to the left of $h(a(u), a(x))$. Hence, if $a(z)$ is to the right of $h(a(u), a(x))$, then extension region $R(u)$ for f_1 and edge $(a(x), a(z))$ are separated by $h(a(u), a(x))$, hence they do not intersect. Suppose that $a(z)$ is to the left of $h(a(u), a(x))$. Suppose, for a contradiction, that extension region $R(u)$ for f_1 crosses edge $(a(z), a(x))$ and denote by r any such an intersection point. Since edges $(a(u), a(x))$, $(a(v), a(x))$, and $(a(z), a(x))$ appear in this clockwise order around $a(x)$, it follows that edge $(a(v), a(x))$ lies in the wedge with an angle smaller than 180° delimited by $h(a(x), a(u))$ and $h(a(x), a(z))$. However, if $a(v)$ lies outside triangle $(a(x), a(u), r)$, then edge $(a(v), a(x))$ intersects extension region $R(u)$ for f_1 (see Figure 19(a)), which we already proved not to be the case, while if $a(v)$ lies inside triangle $(a(x), a(u), r)$, then edge $(a(z), a(v))$ crosses extension region $R(u)$ for f (see Figure 19(b)), which contradicts the assumption that Γ_o is an extensible drawing.



(a) (b)

Figure 19: Illustration for the proof that extension region $R(u)$ for f_1 does not intersect edge $(a(z), a(x))$.

- It can be proved analogously that extension region $R(v)$ for f_1 does not intersect edge $(a(z), a(x))$.

- To ensure that extension region $R(x)$ for f_1 does not intersect edges $(a(u), a(x))$, $(a(v), a(x))$, and $(a(z), a(x))$, it suffices to choose the point p around which $S(u, v, x)$ lies so that p does not lie on segments $\overline{a(u)a(x)}$, $\overline{a(v)a(x)}$, and $\overline{a(z)a(x)}$, and to choose $S(u, v, x)$ small enough.

Property 2. Figure 20 illustrates the structure of the proof that Γ satisfies Property 2 of Definition 3. We prove that no two extension regions for f_1 intersect, except on their borders, unless they both comprise central region $S(u, v, x)$. Analogous proofs for f_2 and f_3 can be exhibited.

Property 2 (for f_1):

- $R(u, v)$ intersects no other extension region for f_1
- $R(u)$, $R(v)$, and $R(x)$ do not intersect $R(u, x)$ and $R(v, x)$
- $R(u, x)$ does not intersect $R(v, x)$
    - $\sigma(a(x), S(v, x))$ does not intersect $\sigma(a(x), S(u, x))$
    - $\sigma(a(x), S(v, x))$ does not intersect $\sigma(a(u), S(u, x))$
    - $\sigma(a(x), S(u, x))$ does not intersect $\sigma(a(v), S(v, x))$
    - $\sigma(a(u), S(u, x))$ does not intersect $\sigma(a(v), S(v, x))$
        - if $a(u)$ is to the left of $h(u, x)$
        - if $a(v)$ is to the right of $h(v, x)$
        - if $a(u)$ is to the right of $h(u, x)$ and $a(v)$ is to the left of $h(v, x)$
- $R(u)$, $R(v)$, and $R(x)$ do intersect each other

Figure 20: Structure of the proof that Γ satisfies Property 2 of Definition 3.

- Extension region $R(u, v)$ for f_1 does not intersect any of the other extension regions for f_1, given that the latter ones are inside region $R(u) \cup R(v) \cup R(z)$ (where $R(u)$, $R(v)$, and $R(z)$ are extension regions for f), given that extension region $R(u, v)$ for f_1 coincides with extension region $R(u, v)$ for f, and given that Γ_o is an extensible drawing.

- Extension regions $R(u)$, $R(v)$, and $R(x)$ for f_1 do not intersect extension regions $R(u, x)$ and $R(v, x)$ by construction. Namely, the point p around which region $S(u, v, x)$ is drawn is chosen in such a way that segments $\overline{pa(u)}$, $\overline{pa(v)}$, and $\overline{pa(x)}$ intersect neither $R(u, x)$ nor $R(v, x)$ (property (v) in the definition of p); hence, if $S(u, v, x)$ is small enough, extension regions $R(u)$, $R(v)$, and $R(x)$ intersect neither $R(u, x)$ nor $R(v, x)$.

- We prove that extension regions $R(u, x)$ and $R(v, x)$ do not intersect each other. By definition, $R(u, x) \equiv \sigma(a(u), S(u, x)) \cup \sigma(a(x), S(u, x))$ and $R(v, x) \equiv \sigma(a(v), S(v, x)) \cup \sigma(a(x), S(v, x))$. Hence, it suffices to prove that each of $\sigma(a(u), S(u, x))$ and $\sigma(a(x), S(u, x))$ does not intersect any of $\sigma(a(v), S(v, x))$ and $\sigma(a(x), S(v, x))$. Denote by p_u and p_v the intersection points of $S(u, x)$ and $S(v, x)$ with edges (u, x) and (v, x), respectively.
    - Regions $\sigma(a(x), S(v, x))$ and $\sigma(a(x), S(u, x))$ do not intersect each other, provided that $S(u, x)$ and $S(v, x)$ are small enough. In fact,

$a(x)$, p_u, and p_v are not collinear, given that p_u is a point of (u, x), p_v is a point of (v, x), and given that Γ is a c-planar drawing.

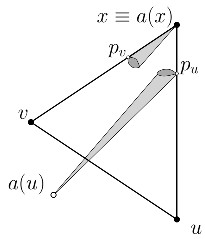


Figure 21: Regions $\sigma(a(x), S(v, x))$ and $\sigma(a(u), S(u, x))$ do not intersect.

- We prove that regions $\sigma(a(x), S(v, x))$ and $\sigma(a(u), S(u, x))$ do not intersect each other. Segment $\overline{a(u)p_u}$ does not intersect segment $\overline{a(x)p_v}$, as otherwise the wedge with an angle smaller than 180° and delimited by $h(x, u)$ and $h(x, a(u))$ would contain vertex v, thus violating Lemma 4. Therefore, it suffices to choose $S(u, x)$ and $S(v, x)$ small enough to ensure that $\sigma(a(x), S(v, x))$ and $\sigma(a(u), S(u, x))$ do not intersect (see Figure 21).
- The proof that regions $\sigma(a(x), S(u, x))$ and $\sigma(a(v), S(v, x))$ do not intersect each other is analogous to the previous one.
- We prove that regions $\sigma(a(u), S(u, x))$ and $\sigma(a(v), S(v, x))$ do not intersect each other. If segment $\overline{a(u)p_u}$ does not intersect segment $\overline{a(v)p_v}$, then it suffices to choose $S(u, x)$ and $S(v, x)$ small enough to ensure that $\sigma(a(u), S(u, x))$ and $\sigma(a(v), S(v, x))$ do not intersect. Assume, for a contradiction, that $\overline{a(u)p_u}$ intersects $\overline{a(v)p_v}$.

  Suppose that $a(u)$ lies to the left of $h(u, x)$. Refer to Figure 22. Then vertex $a(u)$ lies in the wedge with an angle smaller than 180° and delimited by $h(x, u)$ and $h(x, v)$, as otherwise the wedge with an angle smaller than 180° and delimited by $h(x, u)$ and $h(x, a(u))$ would contain v, thus violating Lemma 4. If $a(v)$ lies to the left of $h(a(u), x)$, then $\overline{a(u)p_u}$ and $\overline{a(v)p_v}$ are separated by $h(a(u), x)$, hence they do not intersect. Assume that $a(v)$ lies to the right of $h(a(u), x)$. If $a(v)$ is to the right of $h(v, x)$, then the wedge with an angle smaller than 180° and delimited by $h(x, v)$ and $h(x, a(v))$ contains $a(u)$, thus violating Lemma 4. Finally, if $a(v)$ is to the left of $h(v, x)$, then $\overline{a(u)p_u}$ and $\overline{a(v)p_v}$ are separated by $h(v, x)$, hence they do not intersect.

  The case in which $a(v)$ lies to the right of $h(v, x)$ leads to a contradiction as in the previous case.

  Finally, if $a(u)$ lies to the right of $h(u, x)$ and $a(v)$ lies to the left of $h(v, x)$, then both $a(u)$ and $a(v)$ lie to the left of $h(v, x)$ and to the right of $h(u, x)$, given that segments $\overline{a(u)p_u}$ and $\overline{a(v)p_v}$ intersect each

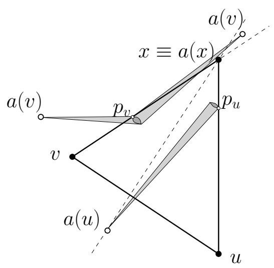


Figure 22: Regions $\sigma(a(u), S(u,x))$ and $\sigma(a(v), S(v,x))$ do not intersect each other if $a(u)$ lies to the left of $h(u,x)$.

other. However, this implies that the wedge with an angle smaller than 180° and delimited by $h(x,u)$ and $h(x,a(u))$ contains vertex $a(v)$, thus violating Lemma 4.

- Finally, extension regions $R(u)$, $R(v)$, and $R(x)$ for f_1 pairwise intersect each other, however they all comprise central region $S(u,v,x)$.

Property 3. Figure 23 illustrates the structure of the proof that Γ satisfies Property 3 of Definition 3. We prove that no extension region for f_1 intersects an extension region for a face of G different from f_1. Analogous proofs for f_2 and f_3 can be exhibited. Since all the extension regions for f_1 are inside extension regions for f, we have that no extension region for f_1 intersects an extension region for a face of G different from f_2 and f_3.

Property 3 (for f_1):

- no extension region for f_1 intersects any extension region for a face different from f_2 and f_3
- no extension region for f_1 intersects any extension region for f_2
    - extension region $R(u,v)$ for f_1 does not intersect any extension region for f_2
    - no extension region for f_1 intersects extension region $R(v,z)$ for f_2
    - extension region $R(v,x)$ for f_1 does not intersect extension region $R(v,x)$ for f_2
    - extension region $R(u,x)$ for f_1 does not intersect extension region $R(v,x)$ for f_2
    - extension region $R(v,x)$ for f_1 does not intersect extension region $R(z,x)$ for f_2
    - extension region $R(u,x)$ for f_1 does not intersect extension region $R(z,x)$ for f_2
    - extension regions $R(u)$, $R(v)$, and $R(x)$ for f_1 do not intersect any extension region for f_2
    - extension regions $R(v)$, $R(z)$, and $R(x)$ for f_2 do not intersect any extension region for f_1
- no extension region for f_1 intersects any extension region for f_3

Figure 23: Structure of the proof that Γ satisfies Property 3 of Definition 3.

We prove that no extension region for f_1 intersects an extension region for f_2. The proof that no extension region for f_1 intersects an extension region for f_3 is analogous.

- Extension region $R(u,v)$ for f_1 coincides with extension region $R(u,v)$ for f, hence it does not intersect any extension region for f_2, given that the latter extension regions are all contained in the extension regions $R(v, z)$, $R(v)$, and $R(z)$ for f, and given that Γ_o is an extensible drawing.

- It can be proved analogously that no extension region for f_1 intersects extension region $R(v, z)$ for f_2.

- Extension region $R(v, x)$ for f_1 does not intersect extension region $R(v, x)$ for f_2 by construction (as observed during the algorithm's description).

- We prove that extension region $R(u, x)$ for f_1 does not intersect extension region $R(v, x)$ for f_2. It is possible to choose side region $S(v, x)$ for f_2 small enough and the intersection points of edge (v, x) with side regions $S(v, x)$ for f_1 and $S(v, x)$ for f_2 close enough so that extension region $R(u, x)$ for f_1 intersects extension region $R(v, x)$ for f_2 if and only if it intersects extension region $R(v, x)$ for f_1; further, we already proved that extension regions $R(u, x)$ and $R(v, x)$ for f_1 do not intersect each other.

- It can be proved analogously that extension region $R(v, x)$ for f_1 does not intersect extension region $R(z, x)$ for f_2 and that extension region $R(u, x)$ for f_1 does not intersect extension region $R(z, x)$ for f_2. The latter proof involves extension regions for f_3: Extension region $R(u, x)$ for f_1 intersects extension region $R(z, x)$ for f_2 only if extension region $R(u, x)$ for f_3 intersects extension region $R(z, x)$ for f_3, which we already proved not to be the case.

- It remains to prove that extension regions $R(u)$, $R(v)$, and $R(x)$ for f_1 do not cross any extension region for f_2 and that extension regions $R(v)$, $R(z)$, and $R(x)$ for f_2 do not cross any extension region for f_1.

  Consider the region R_1 defined as $\mathbb{R}^2 - R(u, v) - R(u, x) - R(v, x)$, where extension regions $R(u, v)$, $R(u, x)$, and $R(v, x)$ for f_1 are here meant to be open regions. Observe that R_1 is composed of two regions, one bounded region "inside" $R(u, v) \cup R(u, x) \cup R(v, x)$, that we denote by R_1^b, and one unbounded region "outside" $R(u, v) \cup R(u, x) \cup R(v, x)$. See Figure 24. Since no two extension regions for f_1 intersect and since central region $S(u, v, x)$ is internal to (u, v, x), extension regions $R(u)$, $R(v)$, and $R(x)$ for f_1 are contained inside R_1^b.

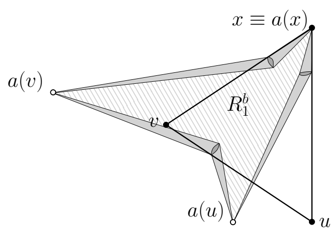


Figure 24: Region R_1^b.

Analogously, the region R_2 defined as $\mathbb{R}^2 - R(v,z) - R(v,x) - R(z,x)$, where extension regions $R(v,z)$, $R(v,x)$, and $R(z,x)$ for f_2 are open regions, is composed of a bounded region R_2^b and of one unbounded region. Extension regions $R(v)$, $R(z)$, and $R(x)$ for f_2 are contained inside R_2^b.

It follows that one of the extension regions for f_1 crosses one of the extension regions for f_2 only if one of the extension regions $R(u,v)$, $R(u,x)$, and $R(v,x)$ for f_1 crosses one of the extension regions $R(v,z)$, $R(v,x)$, and $R(z,x)$ for f_2, which we already proved not to be the case.

Property 4. We prove that extension regions $R(u,v)$, $R(v,x)$, and $R(x,u)$ for f_1 do not intersect clusters other than $\mu(u)$ and $\mu(v)$, other than $\mu(v)$ and $\mu(x)$, and other than $\mu(x)$ and $\mu(u)$, respectively. Analogous proofs hold for the extension regions for f_2 and for f_3. Extension regions $R(u,v)$, $R(v,x)$, and $R(x,u)$ lie inside extension regions for f, hence they do not intersect any cluster other than $\mu(u)$, $\mu(v)$, $\mu(z)$, and $\mu(x)$, since Γ_o is an extensible drawing. Extension region $R(u,v)$ for f_1 does not intersect $\mu(z)$, given that extension region $R(u,v)$ for f_1 coincides with extension region $R(u,v)$ for f and given that Γ_o is an extensible drawing; moreover, $R(u,v)$ does not intersect $\mu(x)$, given that $\mu(x)$ is inside $S(u,v,z)$ and that Γ_o is an extensible drawing. Extension region $R(u,x)$ for f_1 intersects neither $\mu(v)$ nor $\mu(z)$, given that $R(u,x)$ is inside extension region $R(u)$ for f and given that Γ_o is an extensible drawing. Analogously, extension region $R(v,x)$ for f_1 intersects neither $\mu(u)$ nor $\mu(z)$.

Property 5. We prove that extension regions $R(u)$, $R(v)$, and $R(x)$ for f_1 do not intersect clusters other than $\mu(u)$, other than $\mu(v)$, and other than $\mu(x)$, respectively. Analogous proofs hold for the extension regions for f_2 and for f_3. Extension regions $R(u)$, $R(v)$, and $R(x)$ lie inside extension regions for f, hence they do not intersect any cluster other than $\mu(u)$, $\mu(v)$, $\mu(z)$, and $\mu(x)$, since Γ_o is an extensible drawing. Extension region $R(u)$ for f_1 intersects neither $\mu(v)$ nor $\mu(z)$, given that it is inside extension region $R(u)$ for f and given that Γ_o is an extensible drawing; furthermore, it does not intersect $\mu(x)$, given that such a cluster is drawn as an arbitrarily small region surrounding x and given that x is outside extension region $R(u)$ for f_1, by construction. Analogously, extension region $R(v)$ for f_1 intersects neither $\mu(u)$, $\mu(z)$, nor $\mu(x)$. Finally, extension region $R(x)$ does not intersect $\mu(u)$, $\mu(v)$, or $\mu(z)$, given that $R(x)$ is inside $S(u,v,z)$ and given that Γ_o is an extensible drawing.

It follows that Γ is an extensible drawing. By construction, the outer face of C is drawn as Γ_o, hence Γ completes Γ_o, which concludes the proof. $\square$

Base Case 3: *G contains more than one internal vertex, does not contain any separating triangle, and does not contain any internal edge (x,y) with $\mu(x) = \mu(y)$.*

Note that if u, v, and z and their incident edges are removed from G, the resulting graph G' is biconnected. Indeed, such a graph is connected because G is maximal and G_o is a cycle of three vertices. If G' is an edge (a,b), then, by the maximality of G, either (u,v,a) or (u,v,b) is a separating triangle, where u and v are two vertices of G_o. If G' is not biconnected and has more than two vertices, then it contains a cut-vertex c. By the maximality of G, there exists a separating triangle (u, v, c), where u and v are two vertices of G_o. It follows that G' is biconnected. Denote by $\mathcal{C}' = (s_{uz}, u_1, u_2, \dots, u_U, s_{uv}, v_1, v_2, \dots, v_V, s_{vz}, z_1, z_2, \dots, z_Z)$ the cycle delimiting the outer face of G', where vertices $s_{uz}, u_1, u_2, \dots, u_U, s_{uv}$ are neighbors of u, vertices $s_{uv}, v_1, v_2, \dots, v_V, s_{vz}$ are neighbors of v, and vertices $s_{vz}, z_1, z_2, \dots, z_Z, s_{uz}$ are neighbors of z. Observe that no vertex of G' is adjacent to all of u, v, and z, as otherwise G would contain a separating triangle. It follows that vertices s_{uv}, s_{vz}, and s_{uz} are all distinct.

In the following we describe how to construct an extensible drawing of C completing Γ_o. This is accomplished in several steps. We first describe how to draw $\mathcal{C}'$; after that, we show how to draw the vertices of G' not in $\mathcal{C}'$, thus completing the drawing of G' and hence the drawing of G. Then we describe how to draw the vertices of A different from $a(u)$, $a(v)$, and $a(z)$, thus completing the drawing of A. Then we show how to draw extension regions for the faces of G, and finally we show how to draw the clusters in T different from $\mu(u)$, $\mu(v)$, and $\mu(z)$, thus completing the construction of an extensible drawing of C.

First, we draw $\mathcal{C}'$. Suppose that $\mu(u)$, $\mu(v)$, and $\mu(z)$ are distinct. Refer

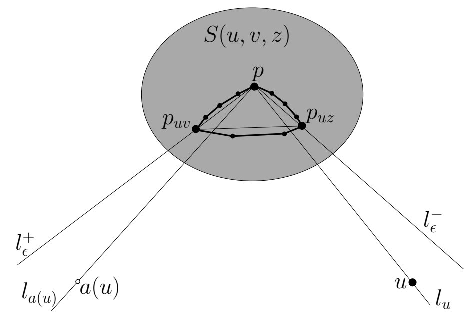


Figure 25: Drawing cycle $\mathcal{C}'$, if $\mu(u)$, $\mu(v)$, and $\mu(z)$ are all different.

to Figure 25. Consider any point p in $S(u, v, z)$. Suppose, w.l.o.g. up to a reflection of the drawing, that the clockwise rotation around p bringing $h(p, u)$ to coincide with $h(p, a(u))$ is smaller than 180°. Let l_ϵ^+ (resp. l_ϵ^-) be the half-line starting at p obtained by clockwise rotating $h(p, a(u))$ by ϵ degrees (resp. by counter-clockwise rotating $h(p, u)$ by ϵ degrees), for some arbitrarily small $\epsilon > 0$. Choose points p_{uv} on l_ϵ^+ and p_{uz} on l_ϵ^- arbitrarily close to p in such a way that segment $\overline{p_{uv}p_{uz}}$ crosses both segment $\overline{pu}$ and segment $\overline{pa(u)}$. Draw $\mathcal{C}'$ as a strictly-convex polygon such that s_{uv} is in p_{uv}, s_{uz} is in p_{uz}, s_{vz} is in p, the slopes of the edges of $\mathcal{C}'$ incident to u_i, for each $1 \leqslant i \leqslant U$ (resp. to v_i, for each $1 \leqslant i \leqslant V$, resp. to z_i, for each $1 \leqslant i \leqslant Z$), are arbitrarily close to the one of segment $\overline{p_{uv}p_{uz}}$ (resp. $\overline{p_{uv}p}$, resp. $\overline{p_{uz}p}$). Denote by $\Gamma_{\mathcal{C}'}$ the resulting drawing.

Next, suppose that $\mu(u) = \mu(v) \neq \mu(z)$ (the cases in which $\mu(u) = \mu(z) \neq \mu(v)$ and $\mu(v) = \mu(z) \neq \mu(u)$ can be treated analogously). Refer to Figure 26. Consider any point p in $S(z, u)$. Consider any arbitrarily small segment $\overline{p_u p_v}$

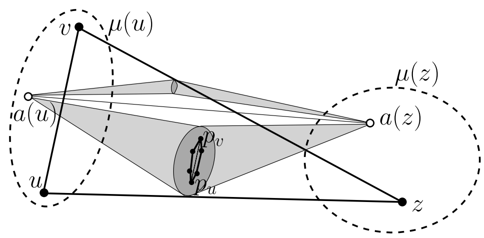


Figure 26: Drawing cycle $\mathcal{C}'$, if $\mu(u) = \mu(v) \neq \mu(z)$.

parallel to edge (u, v) and containing p. Draw $\mathcal{C}'$ as a strictly-convex polygon arbitrarily close to $\overline{p_u p_v}$ such that the slope of every edge in $\mathcal{C}'$ is arbitrarily close to the one of $\overline{p_u p_v}$, such that s_{uz} is mapped to p_u, and such that s_{vz} is mapped to p_v. Denote by $\Gamma_{\mathcal{C}'}$ the resulting drawing.

Second, we draw the vertices of G' not in $\mathcal{C}'$. Since $\Gamma_{\mathcal{C}'}$ is strictly-convex, a drawing $\Gamma_{G'}$ of G' having $\Gamma_{\mathcal{C}'}$ as outer face always exists (see, e.g., [25]). Let Γ_G be the straight-line drawing of G obtained by combining $\Gamma_{G'}$ and Γ_o.

Third, we draw the vertices of A different from $a(u)$, $a(v)$, and $a(z)$. Observe that, for any vertex x of G', there exists no edge (x, y) in G such that $\mu(x) = \mu(y)$, by assumption and since every vertex of G' is an internal vertex of G. An even stronger condition in fact holds: for any vertex x of G', there exists no vertex y in G such that $\mu(x) = \mu(y)$. This comes from the fact that there exists no edge (x, y) in G such that $\mu(x) = \mu(y)$ and from the fact that $C(G, T)$ is c-connected, by Lemma 1. For each vertex x of G', draw $a(x)$ at the same point where x is drawn in Γ_G.

Fourth, we draw extension regions for the faces of G.

We start by drawing extension regions for the faces of G that are not incident to any of u, v, and z. For each face $f = (x, y, t)$ of G such that $\{x, y, t\} \cap \{u, v, z\} = \emptyset$, side region $S(x, y)$ (resp. $S(y, t)$, resp. $S(t, x)$) for f is drawn as an arbitrarily small region inside f touching (x, y) (resp. (y, t), resp. (t, x)); central region $S(x, y, t)$ is drawn as follows: Consider a point p such that p lies inside f and segments $\overline{px}$, $\overline{py}$, and $\overline{pt}$ do not intersect any of the extension regions $R(x, y)$, $R(y, t)$, and $R(t, x)$ for f; such a point always exists provided that side regions $S(x, y)$, $S(y, t)$, and $S(t, x)$ are small enough; then $S(x, y, t)$ is any arbitrary small convex region surrounding p.

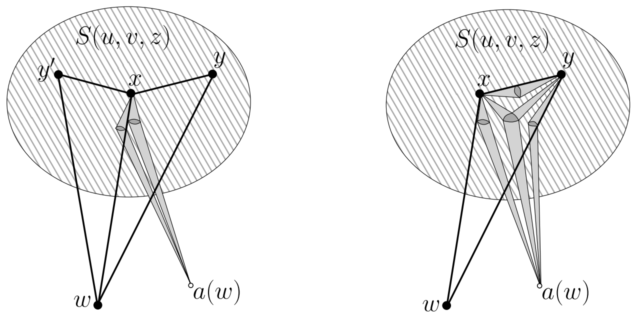
We now draw extension regions for the remaining faces of G (each of which is incident to one or two internal vertices of G).

- For each edge (x, y) incident to the outer face of G (and hence incident to an internal face (x, y, w) of G, where $w \notin \{u, v, z\}$), draw side region $S(x, y)$ for (x, y, w) exactly as side region $S(x, y)$ for f in Γ_o.

- For each edge (x, y) incident to the outer face of G' (and hence incident to an internal face (x, y, w) of G, where $w \in \{u, v, z\}$), draw side region

$S(x,y)$ for (x,y,w) as an arbitrarily small region inside $S(u,v,z)$ (inside $S(z,u)$ if $\mu(u)=\mu(v)$), inside (x,y,w), and touching (x,y).

- For each edge (x,w) such that x is an internal vertex of G and $w \in \{u,v,z\}$, draw extension regions $R(x,w)$ for the two faces incident to (x,w), say (x,w,y) and (x,w,y'), as follows (see Figure 27(a)). Assume, w.l.o.g., that x, w, and y occur in this counter-clockwise order around face (x,w,y) (and hence x, w, and y' occur in this clockwise order around face (x,w,y')). Suppose that $a(w)$ is to the left of $h(x,w)$, the case in which it is to the right being analogous. Then side region $S(x,w)$ for face (x,w,y) is drawn as a small region inside $S(u,v,z)$ (inside $S(z,u)$ if $\mu(u)=\mu(v)$), inside (x,w,y), and touching (x,w) in a point p very close to x. Side region $S(x,w)$ for face (x,w,y') is drawn as a small region inside $S(u,v,z)$ (inside $S(z,u)$ if $\mu(u)=\mu(v)$), inside (x,w,y'), and touching (x,w) in a point q very close to x such that q is farther from x than p. Observe that this implies that extension regions $R(x,w)$ for (x,w,y) and $R(x,w)$ for (x,w,y') do not intersect.



(a) (b)

Figure 27: (a) Drawing extension regions $R(x,w)$ for the faces (x,w,y) and (x,w,y'). (b) Drawing extension region $R(x,y,w)$ for face (x,y,w).

- For each face (x,y,w) such that x and y are internal vertices of G and such that $w \in \{u,v,z\}$, draw central region $S(x,y,w)$ for (x,y,w) as follows (see Figure 27(b)). Consider a point p that lies inside $S(u,v,z)$ (inside $S(z,u)$ if $\mu(u)=\mu(v)$), inside (x,y,w), and such that segments $\overline{pa(x)}$, $\overline{pa(y)}$, and $\overline{pa(w)}$ do not intersect any of the extension regions $R(x,y)$, $R(x,w)$, and $R(y,w)$ for (x,y,w). Such a point always exists provided that side regions $S(x,y)$, $S(x,w)$, and $S(y,w)$ for (x,y,w) are small enough. Then $S(x,y,w)$ is any arbitrarily small convex region surrounding p.

- For each face (x,y,w) such that x is an internal vertex of G and such that $y,w \in \{u,v,z\}$, draw central region $S(x,y,w)$ for (x,y,w) as follows: Consider a point p that lies inside $S(u,v,z)$ (inside $S(z,u)$ if $\mu(u)=\mu(v)$), inside (x, y, w), and such that segments $\overline{pa(x)}$, $\overline{pa(y)}$, and $\overline{pa(w)}$ do not intersect any of the extension regions $R(x, y)$, $R(x, w)$, and $R(y, w)$ for (x, y, w). Such a point always exists provided that side regions $S(x, y)$ and $S(x, w)$ are small enough. Then $S(x, y, w)$ is any arbitrarily small convex region surrounding p.

Fifth, for each vertex x of G, we draw cluster $\mu(x)$ as an arbitrarily small convex region surrounding x. Again, recall that for any vertex x of G', cluster $\mu(x)$ does not contain any vertex $y \neq x$, because there exists no edge (x, y) in G such that $\mu(x) = \mu(y)$ and because $C(G, T)$ is c-connected, by Lemma 1.

Denote by Γ the resulting drawing.

Figure 28 shows the construction of an extensible drawing of C in Base Case 3 if $\mu(u)$, $\mu(v)$, and $\mu(z)$ are all different.

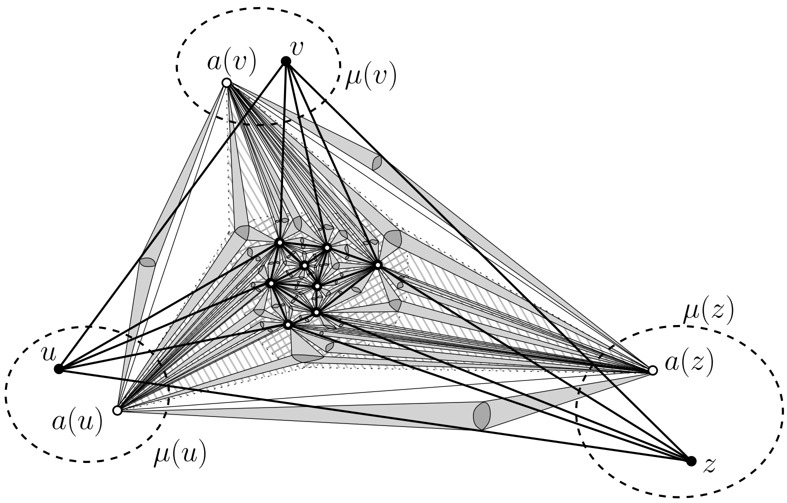


Figure 28: Base Case 3, if $\mu(u)$, $\mu(v)$, and $\mu(z)$ are all different.

Lemma 7 Γ *is an extensible drawing of* C *completing* Γ_o.

Proof: We prove the statement in the case in which $\mu(u)$, $\mu(v)$, and $\mu(z)$ are all different. The cases in which $\mu(u) = \mu(v) \neq \mu(z)$, or $\mu(u) = \mu(z) \neq \mu(v)$, or $\mu(v) = \mu(z) \neq \mu(u)$ are analogous and simpler. In the following we will prove that: (i) Γ is a c-planar drawing; (ii) Γ is a convex drawing; (iii) Γ is a multilayer drawing; (iv) the extension regions satisfy the properties of Definition 2; and (v) Γ is an extensible drawing. Due to the similarity of this proof to the one of Lemma 6, some arguments are not presented here.

C-planar drawing: To prove the planarity of Γ_G, it suffices to show that the edges incident to u, v, and z do not intersect each other and do not intersect the edges of $\mathcal{C}'$. Namely, the internal edges of G' do not intersect each other and do not intersect the edges of $\mathcal{C}'$, given that cycle $\mathcal{C}'$ is drawn as a strictly-convex polygon and that a planar drawing of an internally-triangulated planar graph always exists for any strictly-convex drawing of its outer face (see, e.g., [25]).

All the edges incident to u (to v, to z) and internal to G are inside triangle $\Delta_u = (u, p_{uv}, p_{uz})$ (resp. $\Delta_v = (v, p, p_{uv})$, resp. $\Delta_z = (z, p_{uz}, p)$). Moreover, the only edges of $\mathcal{C}'$ that are inside triangle Δ_u (Δ_v, Δ_z) are those incident to some vertex u_i (resp. v_i, resp. z_i). Hence, as long as triangles Δ_u, Δ_v, and Δ_z have disjoint interiors, each edge incident to u (resp. to v, resp to z) can possibly intersect only an edge incident to a vertex u_i (resp. v_i, resp. z_i), and each edge incident to a vertex u_i (resp. v_i, resp. z_i) can possibly intersect only an edge incident to a vertex u_i (resp. v_i, resp. z_i) and an edge incident to u (resp. to v, resp. to z). Denote by Δ_{uvz} the triangle (p, p_{uv}, p_{uz}). That triangles Δ_u, Δ_v, and Δ_z have disjoint interiors is trivially implied by the following claim.

Claim 1 *The clockwise orders of the vertices along the border of triangles Δ_u, Δ_v, Δ_z, and Δ_{uvz} in Γ are (u, p_{uv}, p_{uz}), (v, p, p_{uv}), (z, p_{uz}, p), and (p_{uv}, p, p_{uz}), respectively.*

Proof: Refer to Figure 29.

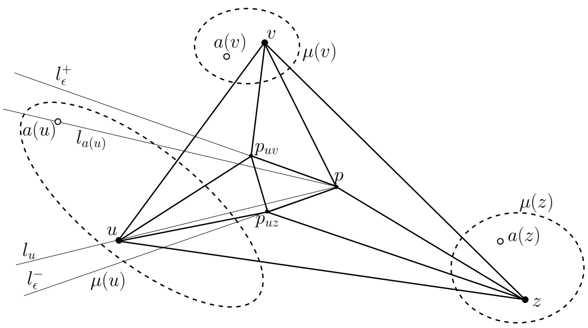


Figure 29: Illustration for the proof of Claim 1.

Clockwise order (u, p_{uv}, p_{uz}) is ensured by the fact that $h(u, p)$ cuts segment $\overline{p_{uv}p_{uz}}$ by leaving p_{uv} to the left and p_{uz} to the right, by construction.

Clockwise order (v, p, p_{uv}) is proved by the following three observations: 1) v does not lie in the wedge with an angle smaller than 180° delimited by $h(p, u)$ and $h(p, a(u))$, by Lemma 4; 2) if v is to the right of $h(u, p)$ and to the right of $h(a(u), p)$, then either vertices u, v, and z are not in this clockwise order around the outer face of G_o, or p is not internal to triangle (u, v, z); 3) hence v is to the left of $h(a(u), p)$, thus if $\epsilon > 0$ is small enough, then v is also to the left of l_ϵ^+ when traversing such a line from p_{uv} to p.

Clockwise order (z, p_{uz}, p) is proved symmetrically to the proof for (v, p, p_{uv}).

Clockwise order (p_{uv}, p, p_{uz}) is ensured by construction. □

It remains to prove that there are no crossings only involving edges incident to u and/or to a vertex u_i (analogous arguments deal with the crossings only involving edges incident to v and/or to a vertex v_i, and crossings only involving edges incident to z and/or to a vertex z_i). However, this comes from the fact that the slopes of the edges of $\mathcal{C}'$ incident to u_i, for each $1 \leqslant i \leqslant U$, are arbitrarily close to the one of segment $\overline{p_{uv}p_{uz}}$, hence the order of the neighbors of u in Γ is $s_{uz}, u_1, u_2, \ldots, u_U, s_{uv}$. This shows that Γ has no edge crossings.

By construction, each cluster $\mu(x)$ containing a vertex x of G' is drawn as an arbitrary small region containing x. It follows that $\mu(x)$ intersects only the edges incident to x. All the edges of G' are in $S(u,v,z)$, hence they do not intersect any of $\mu(u)$, $\mu(v)$, and $\mu(z)$, since Γ_o is an extensible drawing. Each edge incident to u and different from (u,v) and (u,z) is in $\sigma(u, S(u,v,z))$, hence it intersects neither $\mu(v)$ nor $\mu(z)$. Analogously, each edge incident to v and different from (u,v) and (v,z) intersects neither $\mu(u)$ nor $\mu(z)$, and each edge incident to z and different from (u,z) and (v,z) intersects neither $\mu(u)$ nor $\mu(v)$. It follows that Γ has no edge-region crossings.

Any cluster $\mu(x)$ containing a vertex x of G' does not intersect any cluster $\mu(y)$ containing a vertex y of G', since $\mu(x)$ and $\mu(y)$ are arbitrary small regions surrounding x and y, respectively, and does not intersect any of $\mu(u)$, $\mu(v)$, and $\mu(z)$, because $\mu(x)$ is inside $S(u,v,z)$ and such a region has no intersection with $\mu(u)$, $\mu(v)$, and $\mu(z)$, given that Γ_o is an extensible drawing. No two clusters out of $\mu(u)$, $\mu(v)$, and $\mu(z)$ intersect, given that Γ_o is an extensible drawing. It follows that Γ has no region-region crossings, hence it is a c-planar drawing.

Convex drawing: Clusters $\mu(u)$, $\mu(v)$, and $\mu(z)$ are convex, since Γ_o is an extensible drawing, and the clusters containing vertices of G' are convex by construction. It follows that Γ is a straight-line convex drawing.

Multilayer drawing: Vertices $a(u)$, $a(v)$, and $a(z)$ are inside $\mu(u)$, $\mu(v)$, and $\mu(z)$, respectively, since Γ_o is an extensible drawing. Further, for any vertex x of G', $a(x)$ coincides with x and $\mu(x)$ surrounds x; hence $a(x)$ is inside $\mu(x)$.

By construction, each cluster $\mu(x)$ containing a vertex x of G' is drawn as an arbitrary small region containing x, hence $\mu(x)$ intersects only the edges of A incident to $a(x)$. Each edge $(a(x), a(y))$ of A such that $x, y \in G'$ is in $S(u,v,z)$ hence it does not intersect any of $\mu(u)$, $\mu(v)$, and $\mu(z)$, since Γ_o is an extensible drawing. Each edge of A incident to $a(u)$ and different from $(a(u), a(v))$ and $(a(u), a(z))$ is in $R(u)$, hence it does not intersect any of $\mu(v)$ and $\mu(z)$, since Γ_o is an extensible drawing. Analogously, each edge incident to $a(v)$ and different from $(a(u), a(v))$ and $(a(v), a(z))$ does not intersect any of $\mu(u)$ and $\mu(z)$, and each edge incident to $a(z)$ and different from $(a(u), a(z))$ and $(a(v), a(z))$ does not intersect any of $\mu(u)$ and $\mu(v)$. It remains to prove that the drawing of A is planar. However, such a proof can be conducted analogously to the proof that the drawing of G is planar. In particular, the fact that the edges incident to $a(u)$ and to $a(u_i)$ do not intersect is a consequence of the property that the clockwise order of $a(u)$, $a(v)$, and $a(z)$ around any point p of $S(u,v,z)$ is the same as the clockwise order of u, v, and z around p and from the choice of the slopes of the edges of $\mathcal{C}'$. It follows that Γ is a multilayer drawing.

Extension regions and extensible drawing: All the properties that have to be satisfied by the extension regions for the faces of G are trivially satisfied for all the faces of G', that is, for all the faces of G not incident to any outer vertex. Analogously to the proof of Lemma 6, the following facts can be proved: (i) regions $\sigma(x, S(x,y))$ and $\sigma(x, S(x,y,w))$ are inside face (x,y,w), with $\{x,y,w\} \cap \{u,v,z\} \neq \emptyset$, (ii) regions $\sigma(x, S(x,y,w))$, $\sigma(x, S(x,y))$, and $\sigma(x, S(x,w))$ do not intersect any cluster other than $\mu(x)$, and (iii) regions $\sigma(x, S(x,y,w))$, $\sigma(x, S(x,y))$, and $\sigma(x, S(x,w))$ do not intersect regions $\sigma(y, S(x,y,w))$, $\sigma(y, S(x,y))$, and $\sigma(y, S(y,w))$. Finally, the proof that Γ satisfies the properties related to the extension regions as in Definition 3, can be conducted in an analogous way as in the proof of Lemma 6.

It follows that Γ is an extensible drawing. By construction, the outer face of C is drawn as Γ_o, hence Γ completes Γ_o, which concludes the proof. □

6 Inductive Cases

In this section we present the inductive cases for the proof of Theorem 2.

Inductive Case 1: *G contains a separating triangle* (u', v', z'). Denote by $C'_o(G'_o, T'_o)$ the clustered graph such that G'_o is cycle (u', v', z'), and T'_o is the subtree of T whose clusters contain at least one vertex of G'_o. We apply the operation "split along a separating triangle", defined in Section 3, in order to split $C(G,T)$ in two clustered graphs $C^1(G^1, T^1)$ and $C^2(G^2, T^2)$. By Lemma 2, $C^1(G^1, T^1)$ and $C^2(G^2, T^2)$ are maximal c-planar flat clustered graphs with fewer vertices than $C(G,T)$. Again by Lemma 2, C_o and C^1_o are the same clustered graph, and C'_o and C^2_o are the same clustered graph. By induction, for an arbitrary extensible drawing Γ_o of C^1_o, there exists an extensible drawing Γ^1 of C^1 completing Γ_o. Cycle (u', v', z') is a face f of G^1. By definition of extensible drawing, the drawing Γ^2_o of C^2_o in Γ^1 is an extensible drawing. Hence, again by induction an extensible drawing Γ^2 of C^2 can be constructed completing Γ^2_o. Plugging Γ^2 into Γ^1 provides a drawing Γ of C.

Lemma 8 *Γ is an extensible drawing of C completing Γ_o.*

Proof: Let A, A^1, A^2, A^1_o, and A^2_o be the cluster-adjacency graphs of $C(G,T)$, of $C^1(G^1, T^1)$, of $C^2(G^2, T^2)$, of $C^1_o(G^1_o, T^1_o)$, and of $C^2_o(G^2_o, T^2_o)$, respectively. In the following we will prove that: (i) Γ is a c-planar drawing; (ii) Γ is a convex drawing; (iii) Γ is a multilayer drawing; (iv) the extension regions satisfy the properties of Definition 2; and (v) Γ is an extensible drawing.

C-planar drawing: Γ has no edge crossing. Namely, any edge belonging to G^1 (resp. to G^2) does not intersect any edge belonging to G^1 (resp. to G^2) by induction. Further, any edge belonging to G^1 and not belonging to G^2 does not intersect any edge belonging to G^2 and not belonging to G^1 since such edges are separated by cycle (u', v', z').

Further, Γ has no edge-region crossing. Namely, any edge belonging to G^1 (resp. to G^2) does not intersect the boundary of any cluster of T^1 (resp. of T^2) more than once by induction. Further, any edge belonging to G^1 (resp. to G_2) and not belonging to G^2 (resp. to G^1) does not intersect the boundary of any cluster belonging to T^2 (resp. to T^1) and not belonging to T^1 (resp. to T^2), since such an edge and such a cluster are separated by cycle (u', v', z').

Finally, Γ has no region-region crossing. Namely, the boundary of any cluster belonging to T^1 (resp. to T^2) does not intersect the boundary of any cluster belonging to T^1 (resp. to T^2) by induction. Further, the boundary of any cluster belonging to T^1 (resp. to T^2) and not belonging to T^2 (resp. to T^1) does not intersect the boundary of any cluster belonging to T^2 (resp. to T^1) and not belonging to T^1 (resp. to T^2), since such clusters are separated by cycle (u', v', z'). It follows that Γ is c-planar.

Convex drawing: The clusters in T^1 are convex, since Γ^1 is a convex drawing; the clusters in T^2 are convex, since Γ^2 is a convex drawing.

Multilayer drawing: Each vertex $a(x)$ in A^1 is inside cluster $\mu(x)$ in Γ as it is in Γ^1; each vertex $a(x)$ in A^2 is inside cluster $\mu(x)$ in Γ as it is in Γ^2.

We prove that the drawing of A in Γ is planar. No two edges in A^1 (in A^2) intersect as they don't intersect in Γ^1 (resp. in Γ^2). No edge in A^1 intersects an edge in A^2; namely, since Γ^2 is an extensible drawing of G^2 that completes the extensible drawing Γ^2_o of C^2_o in Γ^1, the edges in A_2, except for $(a(u'), a(v'))$, $(a(u'), a(z'))$, and $(a(v'), a(z'))$, are inside the extension regions for the face of G^1 delimited by (u', v', z'). Since Γ^1 is an extensible drawing, no edge in A^1 intersects any extension region for a face of G^1. The statement follows.

No edge in A^1 (in A^2) intersects a cluster in T^1 (resp. in T^2), given that Γ^1 (resp. Γ^2) is an extensible drawing. The clusters in T^2, except for $\mu(u')$, $\mu(v')$, and $\mu(z')$, are inside the extension regions for the face of G^1 delimited by cycle (u', v', z'), hence they do not intersect the edges in A^1. The edges in A^2 different from $(a(u'), a(v'))$, $(a(u'), a(z'))$, and $(a(v'), a(z'))$ are inside the extension regions for the face of G^1 delimited by cycle (u', v', z'), hence they do not intersect the clusters in T^1. It follows that Γ is a multilayer drawing.

Extension regions: The fact that, for any face (x, y, w) of G, central region $S(x, y, w)$ and side regions $S(x, y)$, $S(y, w)$, and $S(w, x)$ are inside (x, y, w) comes from the fact that they are inside (x, y, w) either in Γ^1 or in Γ^2. Analogously, for any face (x, y, w) of G, regions $\sigma(x, S(x, y, w))$, $\sigma(x, S(x, y))$, and $\sigma(x, S(w, u))$ do not intersect each other and do not intersect any cluster in Γ, as they don't intersect each other and do not intersect any cluster in Γ^1 and in Γ^2. Observe that any two regions $\sigma(x, S(x, y, w))$ and $\sigma(x', S(x', y', w'))$ for two different faces (x, y, w) and (x', y', w') of G are disjoint as they lie inside the corresponding faces. Moreover, for every face (x, y, w) of G such that $\mu(x)$, $\mu(y)$, and $\mu(w)$ are all distinct and such that x, y and z come in this clockwise order around the face, and for every point $p \in S(x, y, w)$, segments $\overline{pa(x)}$, $\overline{pa(y)}$, and $\overline{pa(w)}$ are in this clockwise order around p, given that (x, y, w) is also a face of G^1 or of G^2 and given that Γ^1 and Γ^2 are extensible drawings.

Extensible drawing: We now deal with the properties in Definition 3.

1. No extension region for a face of G^1 (of G^2) intersects an edge of A^1 (resp. of A^2), since Γ^1 (resp. Γ^2) is an extensible drawing. No extension region for a face of G^2 intersects an edge of A^1, since all such extension regions are inside the extension regions for the face of G^1 delimited by cycle (u', v', z') and since such extension regions do not intersect edges of A^1, given that Γ^1 is an extensible drawing of C^1. No extension region

for a face of G^1 intersects an edge of A^2, since all such edges are inside the extension regions for the face of G^1 delimited by cycle (u', v', z') and since such extension regions do not intersect other extension regions for the faces of G^1, given that Γ^1 is an extensible drawing of C^1.

2. Since each face f of G is also a face of G^1 or a face of G^2 and since Γ^1 and Γ^2 are extensible drawings of C^1 and C^2, respectively, then no two extension regions for f intersect, except on their borders, unless they are the same region or they both comprise the same central region.

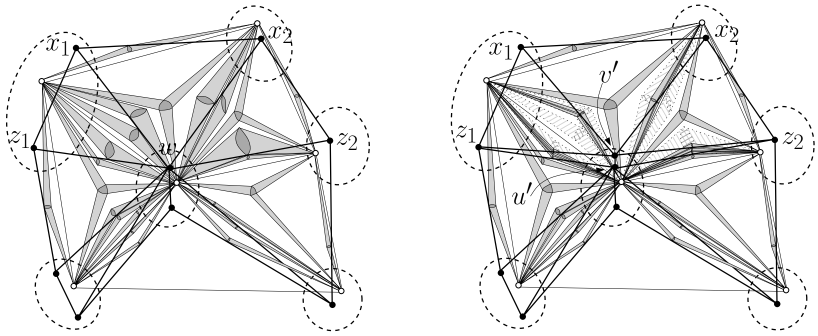
3. No extension region for a face f_1 of G intersects an extension region for a face f_2 of G, with $f_1 \neq f_2$. This comes from the analogous property for Γ^1 and Γ^2, if f_1 and f_2 are both faces of G^1 or are both faces of G^2, respectively. Moreover, no extension region for a face of G^1 intersects an extension region for a face of G^2, since the latter extension regions are inside the extension regions for the face of G^1 delimited by cycle (u', v', z') and since such extension regions do not intersect other extension regions for the faces of G^1, given that Γ^1 is an extensible drawing of C^1.

4. - 5. No extension region for a face of G^1 (of G^2) intersects clusters in T^1 (resp. in T^2), since Γ^1 (resp. Γ^2) is an extensible drawing. No extension region for a face of G^2 intersects clusters in T^1, since all such regions are inside the extension regions for the face of G^1 delimited by cycle (u', v', z') and since such extension regions do not intersect clusters in T^1, given that Γ^1 is an extensible drawing of C^1. No extension region for a face of G^1 intersects clusters in T^2, since all such clusters are inside the extension regions for the face of G^1 delimited by cycle (u', v', z') and since such extension regions do not intersect other extension regions for the faces of G^1, given that Γ^1 is an extensible drawing of C^1.

It follows that Γ is an extensible drawing. By construction, the outer face of C is drawn as Γ_o, hence Γ completes Γ_o, which concludes the proof. $\square$

Inductive Case 2: *G contains no separating triangle and it contains an internal edge (u', v') such that $\mu(u') = \mu(v')$.*

Refer to Figure 30. Since G is maximal and since (u', v') is an internal edge, u' and v' have exactly two common neighbors z_1 and z_2, delimiting internal faces f_1 and f_2 with (u', v'). We apply the operation "contraction of an internal edge", defined in Section 3, in order to transform $C(G, T)$ into a smaller maximal c-planar flat clustered graph $C'(G', T')$. By Lemma 3, $C'(G', T')$ is a maximal c-planar flat clustered graph with fewer vertices than $C(G, T)$; further, C_o and C'_o are the same clustered graph. By induction, for an arbitrary extensible drawing Γ_o of C_o, there exists an extensible drawing Γ' of C' completing Γ_o. Then consider a small disk D centered at w and consider any line l from w to an interior point of the segment between z_1 and z_2. Rename w to u' and insert v' on l, inside D, so that the order of the neighbors of u' in G is the required one. Connect u' and v' to their neighbors.



(a) (b)

Figure 30: (a) An extensible drawing Γ' of C'. (b) The extensible drawing Γ of C obtained from Γ'.

It remains to show how to construct extension regions for the faces of G. The extension regions for each face not incident to v' are drawn as in Γ'. Denote by x_1 the common neighbor of v' and z_1 different from u' and by x_2 the common neighbor of v' and z_2 different from u'. Extension regions $R(u', z_1)$ and $R(z_1, v')$ for face (v', u', z_1) of G and extension region $R(z_1, v')$ for face (v', z_1, x_1) of G are drawn as subsets of extension region $R(w, z_1)$ for face (w, z_1, x_1) of G'. Extension regions $R(v', z_2)$ and $R(z_2, u')$ for face (v', u', z_2) of G and extension region $R(v', z_2)$ for face (v', z_2, x_2) of G are drawn as subsets of extension region $R(w, z_2)$ for face (w, z_2, x_2) of G'. For each neighbor x of v' different from z_1, z_2, and u', extension regions $R(v', x)$ for the two faces incident to edge (v', x) in G are drawn as subsets of the extension region $R(w, x)$ for one of the two faces incident to edge (w, x) in G'. All of the previously described extension regions for the faces of G can in fact be drawn inside the corresponding extension regions for the faces of G' provided that v' is close enough to u', so that edge (v', z_1) cuts the side region $S(w, z_1)$ for face (w, z_1, x_1) of G', so that edge (v', z_2) cuts the side region $S(w, z_2)$ for face (w, z_2, x_2) of G', and so that edge (v', x) cuts one of the two side regions $S(w, x)$ touching edge (w, x). All other extension regions for the faces incident to v' are drawn as the corresponding extension regions for the corresponding faces incident to w in Γ'.

Lemma 9 *Γ is an extensible drawing of C completing Γ_o.*

Proof: In the following we prove that: (i) Γ is a c-planar drawing; (ii) Γ is a convex drawing; (iii) Γ is a multilayer drawing; (iv) the extension regions satisfy the properties of Definition 2; and (v) Γ is an extensible drawing.

C-planar drawing: Γ has no edge crossings. Namely, all the edges that are not incident to v' have exactly the same drawing as in Γ', hence they do not intersect each other since Γ' is an extensible drawing. No two edges incident to v' intersect, as they are adjacent. The construction ensures that the order of the neighbors of v' in Γ is the one specified in G, hence no edge incident to v' intersects an edge between two consecutive neighbors of v'.

Further, Γ has no region-region crossings. Namely, every cluster is represented in Γ by the same region it is represented in Γ'.

Finally, Γ has no edge-region crossings. Namely, consider any edge e of G and any cluster μ in T. If e is not incident to v', then both e and μ have in Γ the same drawing as they have in Γ', hence they do not intersect. If $e = (u', v')$, then e is arbitrarily short, hence it is entirely included inside $\mu(u')$ and it does not intersect the border of any cluster. If $e = (v', z)$ with $z \neq u'$, then the drawing of e is arbitrarily close to the one of edge (w, z) in Γ', hence e crosses the border of each cluster at most once, given that Γ' is an extensible drawing. It follows that Γ is a c-planar drawing.

Convex drawing: Each cluster is represented in Γ by the same convex region it is represented in Γ'.

Multilayer drawing: Each vertex $a(x)$ in A is inside cluster $\mu(x)$ in Γ as it is in Γ'. No two edges in A intersect in Γ as they don't intersect in Γ', thus the drawing of A is planar. No edge of A intersects a cluster of T, as such edge and cluster have the same drawing in Γ as they have in Γ'. It follows that Γ is a multilayer drawing.

Extension regions: For any face (x, y, t) of G with $x, y, t \neq v'$, central region $S(x, y, t)$ and side regions $S(x, y)$, $S(y, w)$, and $S(t, x)$ are inside (x, y, t) in Γ, as they are inside such a face in Γ'. Analogously, regions $\sigma(x, S(x, y, t))$ and $\sigma(x, S(x, y))$ do not intersect each other and do not intersect any cluster in Γ, as they do not intersect each other and do not intersect any cluster in Γ'. The property that, for every face (x, y, w) of G such that $\mu(x)$, $\mu(y)$, and $\mu(w)$ are all distinct and such that x, y and z come in this clockwise order around the face, and for every point $p \in S(x, y, w)$, segments $\overline{pa(x)}$, $\overline{pa(y)}$, and $\overline{pa(w)}$ are in this clockwise order around p, comes from the fact that the same property is satisfied in Γ', from the fact that, for each vertex t in G, $a(t)$ has the same position in Γ and in Γ', and from the fact that each vertex t in G has the same position or two arbitrarily close positions (if $t = v'$) in Γ and in Γ'.

Extensible drawing: We prove that Γ satisfies the properties in Definition 3.

1. No edge of A intersects an extension region in Γ, given that (i) Γ' is an extensible drawing, given that (ii) each edge of A has the same drawing in Γ and in Γ', and given that (iii) each extension region in Γ either has the same drawing as in Γ' or is a subset of an extension region in Γ'.

2. - 3. No two extension regions for faces of G intersect in Γ, given that (i) Γ' is an extensible drawing, (ii) each extension region for a face of G not incident to v' has the same drawing in Γ as in Γ' (hence no two extension regions for faces of G not incident to v' intersect), (iii) each extension region for a face of G incident to v' is a subset of an extension region for a face of G' in Γ' (hence no extension region for a face of G incident to v' intersects an extension region for a face of G not incident to v'), and

(iv) by construction no two extension regions for faces of G in Γ which are drawn inside the same extension region for a face of G' in Γ' intersect (hence no two extension regions for faces of G incident to v' intersect).

4. - 5. No cluster of T intersects an extension region in Γ, given that (i) Γ' is an extensible drawing, (ii) each cluster in T has the same drawing in Γ and in Γ', and (iii) each extension region in Γ either has the same drawing as in Γ' or is a subset of an extension region in Γ'.

It follows that Γ is an extensible drawing. By construction, the outer face of C is drawn as Γ_o, hence Γ completes Γ_o, which concludes the proof. □

By Lemma 5, exactly one among Inductive Cases 1-2 and Base Cases 1-3 applies to $C(G, T)$. This concludes the proof of Theorem 2.

7 Conclusions

We have proved that every flat c-planar clustered graph admits a multilayer drawing. The algorithm we described in this paper uses real coordinates, hence it constructs drawings requiring exponential area to be represented on a screen with a finite resolution rule. However, this drawback is unavoidable, since it has been proved by Feng *et al.* [14] that there exist (flat) clustered graphs requiring exponential area in any straight-line drawing in which clusters are represented by convex regions.

It is an obvious open problem to extend our results to general c-planar clustered graphs. We suspect that our drawing techniques, together with some techniques to decompose non-flat clustered graphs into smaller non-flat clustered graphs presented in [1], might lead to a solution of the problem. However, we defer such an intuition to future research.

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