



## 1-Visibility Representations of 1-Planar Graphs

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### Abstract

A 1-visibility representation of a graph displays each vertex as a horizontal vertex-segment, called a bar, and each edge as a vertical edge-segment between the segments of the vertices, such that each edge-segment crosses at most one vertex-segment and each vertex-segment is crossed by at most one edge-segment. A graph is 1-visible if it has such a representation. 1-visibility is related to 1-planarity where graphs are drawn such that each edge is crossed at most once, and specializes bar 1-visibility where vertex-segments can be crossed many times.

We develop a linear time algorithm to compute a 1-visibility representation of an embedded 1-planar graph in  $\mathcal{O}(n^2)$  area. Hence, every 1-planar graph is 1-visible. Concerning density, both 1-visible and 1-planar graphs of size  $n$  have at most  $4n - 8$  edges. However, for every  $n \geq 7$  there are 1-visible graphs with  $4n - 8$  edges, which are not 1-planar.

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## 1 Introduction

Drawing planar graphs is an important topic in graph theory, combinatorics, and in particular in graph drawing. The existence of straight-line drawings was independently proved by Wagner [34], Steinitz and Rademacher [29], Stein [28] and Fáry [17]. The results of de Fraysseix, Pach and Pollack [8] and Schnyder [27] show that planar graphs admit straight-line grid drawings in quadratic area, which can be computed in linear time.

A *visibility representation* is another way to draw a planar graph. Here the vertices are drawn as horizontal vertex-segments, called bars, and two segments must see each other along a vertical line-of-sight if there is an edge between the respective vertices. This is the weak version of visibility. In the strong version there is a one-to-one correspondence between edges and visibility. Otten and van Wyck [22] showed first of all that every planar graph has a weak visibility representation. A linear time algorithm for constructing it was given independently by Rosenstiehl and Tarjan [26] and by Tamassia and Tollis [32]. The algorithm uses a grid of size at most  $(2n - 5) \times (n - 1)$ , which was improved to  $(\lfloor 4n/3 \rfloor - 2) \times (n - 1)$  by Fan et al. [16]. The weak and strong versions of visibility were characterized in [32, 35].

There are several attempts towards *beyond planar* graphs. Generally, beyond planar graphs extend the planar graphs by regulating edge crossings in some way. The attempts use forbidden minors [25], surfaces of higher genus, or various restrictions on crossings, such as  $k$ -planar [23],  $k$ -quasi-planar [1] or right angle crossing (RAC) graphs [11]. Here, we consider 1-planar graphs which are defined by drawings such that (the Jordan curve of) an edge is crossed at most once. 1-planar graphs were introduced by Ringel [24] and occur when a planar graph and its dual are drawn simultaneously.

The straight-line or rectilinear drawability of 1-planar graphs was first investigated by Eggleton [13]. He settled this problem for outer 1-planar graphs and proved that every outer 1-planar graph has a straight-line drawing. In outer 1-planar graphs all vertices are in the outer face and each edge is crossed at most once. Thomassen [33] generalized this result and proved that an embedded 1-planar graph has a straight-line drawing if and only if it excludes B- and W-configurations, see Figs. 1(a) and 1(b). Then only X-configurations remain for pairs of crossing edges, see Fig. 1(c). The configurations were rediscovered by Hong et al. [19], who also showed that there is a linear time algorithm to convert a 1-planar embedding without B- and W-configurations into a straight-line drawing. Alam et al. [2] showed that every 3-connected 1-planar graph has an embedding with at most one W-configuration in the outer face, and has a straight-line grid drawing in quadratic area with the exception of a single edge in the outer face. Here we add visibility representations. These drawings can be computed in linear time from a given 1-planar embedding as a witness for 1-planarity. Note that 1-planarity testing is NP-hard [21].

There is a close relationship between 1-planar graphs and right angle crossing (RAC) graphs, in which edges must be straight-line and cross at a right angle [11]. 1-planar graphs and RAC graphs have almost the same density, i.e., the

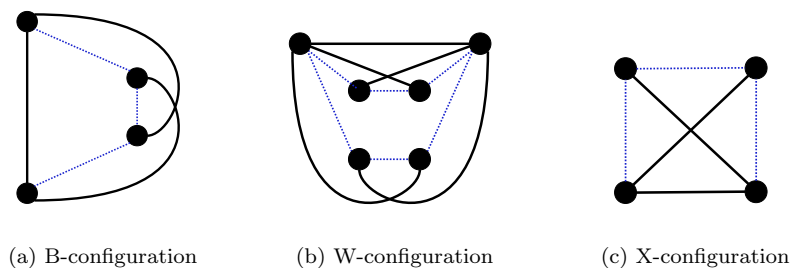


Figure 1: Augmented B-, W- and X-configurations, where the augmentation is drawn with blue dotted lines.

maximal number of edges for graphs of size  $n$ , namely  $4n - 8$  and  $4n - 10$ . Eades and Liotta [12] showed that every maximally dense RAC graph is 1-planar, and that there are RAC graphs which are not 1-planar.

Dean et al. [9] introduced *bar  $k$ -visibility* where the vertices are represented as horizontal bars and bars are allowed to see vertically through at most  $k$  other bars. Thus bar 0-visibility is the common planar visibility, and in bar 1-visibility a bar can be crossed by many visibility lines between other bars. They proved that bar 1-visibility graphs have at most  $6n - 20$  edges, which is a tight bound, and showed that the complete graph  $K_8$  is bar 1-visible, whereas  $K_9$  is not. Our definition of 1-visibility is the restriction of bar 1-visibility, such that each edge-segment crosses at most one bar and each bar is crossed by at most one edge-segment. A graph is called *1-visible* if it admits a 1-visibility representation.

Recently, Sultana et al. [30] showed that some special classes of graphs admit a bar 1-visibility representation, and they conjectured that every 1-planar graph has such a representation. We prove a tightened version of their conjecture and develop a linear time algorithm which converts a 1-planar drawing into a 1-visibility representation. The fact that every 1-planar graph has a bar 1-visibility representation was recently independently obtained by Evans et al. [14], who also studied the weak and strong versions of bar 1-visibility.

Our algorithm uses standard techniques for visibility representations of planar graphs [10, 26, 32], a local transformation to re-insert pairs of crossing edges and a matching to guarantee that each vertex-segment is crossed at most once. It constructs a 1-visibility representation of a graph  $G$  in an area of size at most  $4(2n - 5) \times (n - 1)$  in linear time from a 1-planar embedding of  $G$ .

1-visibility representations have straight-line edges, even for the edges from the *B*- and *W*-configurations, which cannot be drawn straight-line in 1-planar drawings. Moreover, they are drawings with right angle crossings between vertex and edge-segments. As an example, see the so-called extended wheel graph  $XQ_8$  [5] with 8 vertices and 20 edges in Fig. 2, where the visibility representation is obtained by our algorithm.

1-visible graphs seem to be quite close to 1-planar graphs. They have the

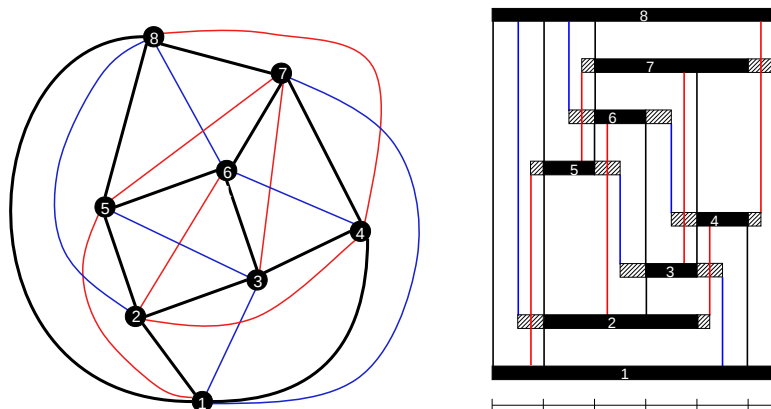


Figure 2: The extended wheel graph  $XQ_8$  (left) and its 1-visibility representation (right). Each red edge crosses a blue one (left) and its edge-segment crosses a vertex-segment (right).

same maximal density with at most  $4n - 8$  edges for graphs of size  $n$ . This is readily seen, since a 1-visible graph consists of a planar subgraph together with at most one crossing edge for all but the two outermost vertices in the 1-visibility representation. By Theorem 1, we obtain a new and simple proof of the maximal density of 1-planar graphs, which was proved first of all by Bodendiek et al. [5], and independently in [15, 23].

However, there are 1-visible graphs with  $4n - 8$  edges for every  $n \geq 7$ , which are not 1-planar, including the complete graph on 7 vertices without one edge  $K_7 - e$ . Hence, the 1-visible graphs properly include the 1-planar graphs, even for maximally dense graphs. Note that  $K_7$  is not 1-visible, since it is too dense, such that  $K_6$  is the largest complete 1-visible graph. Hence, the complete graphs  $K_7$  and  $K_8$  and the density show that 1-visibility is a proper restriction of bar 1-visibility.

Our algorithm and the main result are described in Section 3 and the density result is given in Section 4.

## 2 Preliminaries

Consider simple undirected graphs  $G = (V, E)$  with  $n$  vertices and  $m$  edges. We assume that the graphs are 2-connected, otherwise, the components are treated separately, and are placed next to each other, as proposed in [32].

A *drawing* of a graph is a mapping of  $G$  into the plane such that the vertices are mapped to distinct points and each edge is a Jordan curve between its endpoints. A drawing is *planar* if (the Jordan curves of the) edges do not cross and is *1-planar* if each edge is crossed at most once. In 1-planar drawings

crossings of edges with a common endpoint are excluded.

An embedding  $\mathcal{E}(G)$  of a planar graph  $G$  specifies faces. A *face* is a topologically connected region and is given by a cyclic sequence of edges and vertices that forms its boundary. One of the faces is unbounded and is called *outer face*. Accordingly, a *1-planar embedding*  $\mathcal{E}(G)$  specifies the faces in a 1-planar drawing of a graph  $G$  including the outer face. A 1-planar embedding is a witness for 1-planarity. In particular, it describes the pairs of crossing edges and the face of the planar embedding after the removal of all crossing edges in which a pair of edges cross. A face of  $\mathcal{E}(G)$  is given by a cyclic list of edges and half-edges and their vertices and crossing points. A half-edge is a segment of an edge from a vertex to a crossing point. Each *crossing point* in a 1-planar embedding is incident to four half-edges. If the crossing points are taken as new vertices and the half-edges as edges, then we have the *planarization* of  $\mathcal{E}(G)$ . This structure is used by algorithms operating on 1-planar embeddings, where crossing points always remain as vertices of degree four and may need special treatment.  $\mathcal{E}(G)$  is an embedded planar graph, in which each crossing point is assigned to a face of the embedded planar subgraph after the removal of all crossing points.

A *visibility representation*  $\Gamma$  of a graph  $G = (V, E)$  displays each vertex  $v$  as a horizontal vertex-segment or bar  $\Gamma(v)$ . There is a vertical edge-segment  $\Gamma(e)$  from some point on  $\Gamma(u)$  to some point on  $\Gamma(v)$  if there is an edge  $e = (u, v)$ . Vertex-segments (edge-segments) do not overlap and the endpoints of all segments are grid points.  $\Gamma$  is a bar  $k$ -visibility representation for some  $k \geq 0$  if each edge-segment crosses at most  $k$  vertex-segments. Then an edge can see through up to  $k$  vertices. The case  $k = 0$  is the common (weak) planar visibility.  $\Gamma$  is a 1-visibility representation and  $G$  is called a *1-visible graph* if each edge-segment crosses at most one vertex-segment, and each vertex-segment is crossed by at most one edge-segment. The latter implies a restriction to bar 1-visibility representations, in which vertex-segments can be crossed many times.

We consider the weak version of visibility. In the strong version there is a one-to-one correspondence between edges and visibility [9, 10, 14, 32, 35]. The weak bar  $k$ -visible graphs are exactly the subgraphs of the strong bar  $k$ -visible graphs. The strong versions do not seem appropriate for the definition of beyond planar graphs, since many planar graphs are excluded for 1-visible graphs, e.g., cycles of length at least four, bipartite graphs  $K_{2,k}$  for  $k \geq 3$  and  $n \times m$  grids. In fact, there is no strong 1-visibility representation with a quadrilateral as an inner face.

Next, we recall some facts about 1-planar graphs and their embeddings. The use of embeddings seems crucial, since 1-planarity testing is NP-hard [21] and 1-planar graphs may have different 1-planar embeddings [31]. For example, some planar and some crossing edges can be switched in the  $XQ_8$  graph from Fig. 2.

A 1-planar embedding is *planar maximal* if no further edge can be added without introducing a crossing or multiple edges. A 1-planar embedding can be augmented to a planar maximal embedding. The augmentation can be computed in linear time from the embedding using the planarization and keeping the crossing points as vertices of degree four. Note that the maximality depends

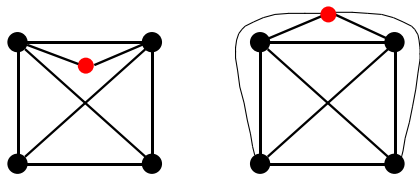


Figure 3: A maximal 1-planar embedding (left) and another maximal 1-planar embedding (right) with the degree two vertex in the outer face.

on the embedding and a different embedding of a graph may give rise to another maximal planar augmentation, as Fig. 3 illustrates.

Planar maximal embeddings have some nice properties, which were established by several authors, e.g., [2, 5, 19].

**Lemma 1** *Let  $\mathcal{E}(G)$  be a planar maximal 1-planar embedding.*

1. *Every crossing induces a  $K_4$  of the end vertices of the crossing edges.*
2. *A face has at most four vertices and at most four crossing points. Every (inner or outer) face is at most a  $k$ -gon with  $k \leq 8$ , in which vertices and crossing points or alternatively half-edge are counted.*

Note that the type of a configuration (B, W or X) depends on the embedding and the choice of the outer face. This observation was used by Alam et al. [2] in their normal form theorem for embeddings of 3-connected 1-planar graphs. Here, a given embedded 1-planar graph is first augmented by planar edges to a planar maximal 1-planar graph and then the embedding is transformed into normal form by local changes in the cyclic order of the neighbors of some vertices. An embedding  $\mathcal{E}(G)$  is in *normal form* if it has at most one augmented W-configuration in the outer face, no augmented B-configuration, and an augmented X-configuration does not contain a vertex inside the boundaries of the quadrangle of its endpoints.

**Lemma 2** *(Normal Form Theorem) [2]*

*Let  $G = (V, E)$  be a 3-connected 1-planar graph and  $\mathcal{E}(G)$  a 1-planar embedding. There is a linear time algorithm to transform  $\mathcal{E}(G)$  into a planar maximal 1-planar embedding of a supergraph  $H = (V, F)$  with  $E \subseteq F$  such that the embedding  $\mathcal{E}(H)$  is in normal form.*

The normal form theorem holds for every 3-connected component of a 1-planar graph  $G$ . Suppose that  $G$  is 2-connected with an embedding  $\mathcal{E}(G)$  with planar maximal 3-connected components in normal form. For every separation pair  $\{u, v\}$  there is a sequence of 3-connected 1-planar graphs  $C_0, \dots, C_{k-1}$  in clockwise order at  $u$ , and each pair of adjacent components  $C_i$  and  $C_{i+1}$  with

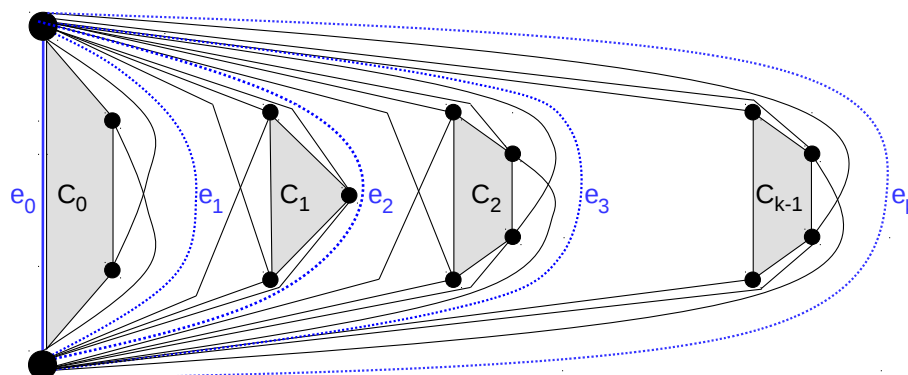


Figure 4: A sequence of planar maximal 1-planar graphs at a separation pair  $\{u, v\}$  and separating edges between the components.

$0 \leq i \leq k - 1$  is separated by a pair of crossing edges from a W- or an X-configuration or both. Otherwise, such components merge to a single planar maximal 3-connected component.

To separate the components at a separation pair  $\{u, v\}$  even further we allow multi-edges and introduce copies  $e_i$  for  $i = 1, \dots, k$  of the edge  $e_0 = (u, v)$  as *separation edges*. The  $i$ -th separation edge  $e_i$  is routed next to a pair of crossing edges which separates  $C_{i-1}$  from  $C_i$ . The outermost separation edge  $e_k$  encloses all components and the multi-edges  $e_0$  and  $e_k$  form the outer face. This situation is depicted in Fig. 4, where the copies of the edge  $e_0 = \{u, v\}$  are drawn dotted and blue. The outermost edge can be omitted if the separation pair is in the outer face and there is no pair of crossing edges from a W-configuration in the outer face.

The steps for the augmentation to 3-connected components in normal form and for the insertion of separation edges take linear time on  $\mathcal{E}(G)$ . Thus we can state.

**Lemma 3** *Every 1-planar embedding  $\mathcal{E}(G)$  can be transformed in linear time into a planar maximal 1-planar embedding  $\mathcal{E}(G')$  of a supergraph with multi-edges  $G'$ , where each 3-connected component of  $\mathcal{E}(G')$  is in normal form and there is a separation edge between adjacent 3-connected components at a separation pair  $\{u, v\}$ .*

For 1-planar graphs, a visibility representation has advantages over a common straight-line drawing. In the next Section we show that all 1-planar graphs can be represented with straight vertical lines in quadratic area but at the expense of bars for the vertices. Straight-line drawings must exclude B- and W-configurations [19, 33]. Each W-configuration induces one edge with a bend, and the sparse maximal 1-planar graphs from Fig. 3 in [6] have a linear number of W-configurations and thus a linear number of edges with a bend. The

algorithm of Alam et al. [2] deals only with 3-connected 1-planar graphs, and Hong et al. [19] showed that a 1-planar embedding can be transformed into a straight-line 1-planar drawing, which preserves the embedding, provided there are no B- and W-configurations. Their algorithm uses the convex drawing algorithm for planar graphs from [7], which needs a high resolution for its numerical computations. There is no stated bound on the area, but it is likely to be exponential, where the expansion of the area is enforced by a sequence of 1-planar graphs at a separation pair, as illustrated by Fig. 4.

### 3 Visibility Representation

In this section we show that every 1-planar graph  $G$  has a 1-visibility representation. The result is obtained by the 1-VISIBILITY algorithm, whose input is an embedding  $\mathcal{E}(G)$  as a witness for 1-planarity. After a planar maximal augmentation it considers each 3-connected component  $C$ , transforms  $C$  into normal form, and separates 3-connected components at a separation pair by separation edges. Then the graph and, in particular, each 3-connected component is planarized by extracting the pairs of crossing edges. This is done via the planarization and then removing all crossing points. The so obtained planar graph  $G_p$  is drawn by a common planar visibility algorithm.  $G_p$  is a spanning subgraph of  $G$ . Thereafter, CROSSING-INSERTION re-inserts each pair of crossing edges in the face from which it was extracted. Finally, the edge-segments of added edges are hidden.

Consider a planar visibility algorithm from [10, 26, 32]. It takes an embedded planar graph and two vertices  $s, t$  in the outer face and directs the edges according to an  $st$ -numbering from  $s$  to  $t$ . Thereafter each vertex  $v$  but  $s, t$  has a neighbor with a smaller and a larger  $st$ -number than itself and two clockwise-consecutive sub-sequences of incoming and outgoing edges, i.e.,  $G$  is bi-modal [26]. Route the edge  $(s, t)$  to the left of the drawing of  $G$ . Then consider the directed dual  $G^*$ , where  $s^*$  is the face to the right of the  $(s, t)$  edge and  $t^*$  is the outer face, and direct its edges according to the  $s^*t^*$ -numbering of  $G^*$ . Recall that  $G$  was extended by separation edges between 3-connected components, which has an impact on  $G^*$ , since a separation edge splits a face and increases the number of faces by one.

Define the *distance*  $\delta(v)$  of a vertex  $v$  by its  $st$ -number [26, 32] or for a more compact drawing [10] by the length of a longest path from  $s$  and accordingly define the dual distance  $\delta^*(f)$  of a face  $f$  of  $G$ . Then  $\delta(s) = 0$ ,  $\delta(t) = h - 1$ ,  $\delta^*(s^*) = 0$  and  $\delta^*(t^*) = w - 1$  for some  $h \leq n$  and  $w \leq 2n - 5$  and the visibility representation is of size  $w \times h$ . The insertion of separation edges does not affect the upper bound of  $2n - 5$ , since for each separation edge  $e_i$  there is at least one missing edge from  $C_i$  to the next component  $C_{i+1}$  in the sequence of 3-connected components, in which pairs of crossing edges are removed. Hence, there are at most  $2n - 4$  faces after the extraction of all crossing edges and the addition of separating edges.

For the compacted version one must take care that the distance of vertices



$b$  and  $d$  of a quadrangle  $f = (a, b, c, d)$  differs if  $a$  and  $c$  are the minimum and maximum distance vertices (bottom and top) and there is an augmented X-configuration. The requirements are met by the  $st$ -number and can otherwise be achieved by a local lifting as in [3], which cost at most one unit in height per lifting. Moreover, if  $\{u, v\}$  is a separation pair with a sequence of 3-connected components  $C_0, \dots, C_{k-1}$  in clockwise order at  $u$  and separation edges  $e_0, \dots, e_k$  and the  $st$ -number of  $u$  is smaller than the  $st$ -number of  $v$ , then the  $st$ -numbering implies that  $\delta(u) < \delta(w) < \delta(v)$  for every vertex  $w$  from any component  $C_i$  and  $\delta^*(e_{i-1}) < \delta^*(f) < \delta^*(e_i)$  if  $f$  is an inner face of  $C_{i-1}$  and  $\delta^*(e_i)$  is the dual distance of the face immediately to the left of  $e_i$ .

For each edge  $e = (u, v)$  let  $left(e)$  ( $right(e)$ ) be the dual distance  $\delta^*(f)$  of the face  $f$  of  $G$  to the left (right) of  $v$  and let  $left(v)$  ( $right(v)$ ) be the least (largest) dual distance of a face incident with  $v$ .

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**Algorithm 1: PLANAR-VISIBILITY**

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**Input:** A 2-connected planar graph  $G$  (with multi-edges) with a planar embedding  $\mathcal{E}(G)$ .

**Output:** A visibility representation  $\Gamma$  of  $G$ .

- 1 Construct an  $st$ -numbering of  $G$  with  $(s, t)$  on the left.
  - 2 Compute the dual graph  $G^*$ .
  - 3 Compute the distance  $\delta(v)$  for all vertices  $v$  of  $G$  and the dual distance  $\delta^*(f)$  for all faces  $f$ .
  - 4 **foreach** vertex  $v$  of  $G$  **do**
  - 5     | draw the horizontal vertex-segment  $\Gamma(v)$  between  $(\delta^*(left(v)), \delta(v))$   
       | and  $(\delta^*(right(v)) - 1, \delta(v))$ .
  - 6 **foreach** edge  $e = (u, v)$  of  $G$  **do**
  - 7     | draw a vertical edge-segment  $\Gamma(e)$  between  $(\delta^*(left(e)), \delta(u))$  and  
       |  $(\delta^*(left(e)), \delta(v))$ .
  - 8 **return**  $\Gamma$
- 

The correctness of PLANAR-VISIBILITY and the linear running time was proved in [10, 26, 32].

We use PLANAR-VISIBILITY to draw 3-connected components  $C_i$  of 1-planar graphs, whose pairs of crossing edges  $(a, c)$  and  $(b, d)$  are first extracted and are then re-inserted in the face they left behind. The normal form embedding and the added separation edge  $e_i$  to the right of  $C_i$  guarantee that each pair of crossing edges has its own face  $f$ , which is a quadrangle by Lemma 1.  $f$  comes from an augmented X-configuration if it is an inner face or is the outer face of a W-configuration and is immediately to the left of the outermost separation edge.

For a face  $f = (a, b, c, d)$  let  $a$  be the bottom in the visibility drawing of PLANAR-VISIBILITY, i.e., the  $y$ -coordinate  $\delta(a)$  is minimal. We call  $f$  a *left-*

*wing* (*right-wing*) if  $\delta(a) < \delta(b) < \delta(c) < \delta(d)$  and  $b, c$  are to the left (right) of  $f$ , and a *diamond* if  $\delta(a) < \delta(b), \delta(d) < \delta(c)$ .  $f$  is a left-wing or right-wing if  $f$  is the outer face or if  $(a, d)$  is a separation edge.

In a 1-visibility representation there are always two options, which segment of the two non top and non bottom vertices of a quadrangle  $f$  is crossed by a re-inserted edge-segment. A maximal bipartite matching determines one vertex per face and guarantees that each vertex-segment is crossed at most once.

The crossing insertions are illustrated in Fig. 5.

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**Algorithm 2: CROSSING-INSERTION**


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**Input:** A visibility representation  $\Gamma$  of a face  $f$  with the vertices  $(a, b, c, d)$  and bottom  $a$ , and a pair of edges  $(a, c)$  and  $(b, d)$  crossing in  $f$ , such that the vertex-segment of  $b$  is crossed by the edge-segment of  $(a, c)$ . (The case where the other inner vertex is crossed is similar).

**Output:** A 1-visibility representation  $\Gamma$  of  $f$  with the additional edges  $(a, c)$  and  $(b, d)$  such that  $\Gamma((a, c))$  crosses  $\Gamma(b)$ .

- 1 **if**  $f$  is a left-wing **then**
  - 2     | extend  $\Gamma(b)$  by 0.5 and  $\Gamma(c)$  by 0.25 units to the right and draw  
       |  $\Gamma((a, c))$  at the  $x$ -coordinate  $\delta^*(f) - 0.75$  and  $\Gamma((b, d))$  at  $\delta^*(f) - 0.5$
  - 3 **if**  $f$  is a right-wing **then**
  - 4     | extend  $\Gamma(b)$  by 0.5 and  $\Gamma(c)$  by 0.25 units to the left and draw  
       |  $\Gamma((a, c))$  at the  $x$ -coordinate  $\delta^*(f) - 0.25$  and  $\Gamma((b, d))$  at  $\delta^*(f) - 0.5$
  - 5 **if**  $f$  is a diamond **then**
  - 6     | extend  $\Gamma(b)$  by 0.5 units to the right and  $\Gamma(d)$  by 0.5 units to the left,  
       | draw  $\Gamma((b, d))$  at the  $x$ -coordinate  $\delta^*(f) - 0.5$  and draw  $\Gamma((a, c))$  at  
       |  $\delta^*(f) - 0.75$  if  $b$  is crossed and at  $\delta^*(f) - 0.25$  if  $d$  is crossed.
  - 7 **return**  $\Gamma$
- 

**Lemma 4** *If a face  $f = (a, b, c, d)$  is drawn by PLANAR-VISIBILITY, then CROSSING-INSERTION adds the pair of crossing edges  $(a, c)$  and  $(b, d)$  inside  $f$  with exactly one vertex-edge segment crossing.*

**Proof:** If  $f$  is a left-wing, then the vertex-segments of  $b$  and  $d$  end at  $\delta^*(f) - 1$  and the edge-segments are at or to the left of  $\delta^*(f) - 2$ . The edge-segment of  $(a, d)$  is at or to the right of  $\delta^*(f)$ . Hence, the extensions of  $\Gamma(b)$  and  $\Gamma(c)$  do not intersect the edge-segment of  $(a, d)$ . The edges  $(a, c)$  and  $(b, d)$  are routed inside  $f$  and induce a crossing of the segments of  $b$  and  $(a, c)$ . The case where  $f$  is a right-wing is symmetric. Then the edge-segment of  $(a, d)$  is at or to the left of  $\delta^*(f) - 1$ , and the edge-segments  $(a, b), (b, c), (c, d)$  are right aligned at  $\delta^*(f)$ . The vertex-segments of  $b$  and  $c$  begin at  $\delta^*(f)$ . If  $f$  is a diamond with  $b$  on the left and  $d$  on the right, then  $\Gamma(b)$  ends at  $\delta^*(f) - 1$  and  $\Gamma(d)$  begins at  $\delta^*(f)$ , and the  $y$ -coordinates of the vertex-segments of  $b$  and  $d$  are different, since  $\delta$  guarantees this property. Again, there is a single vertex-edge segment

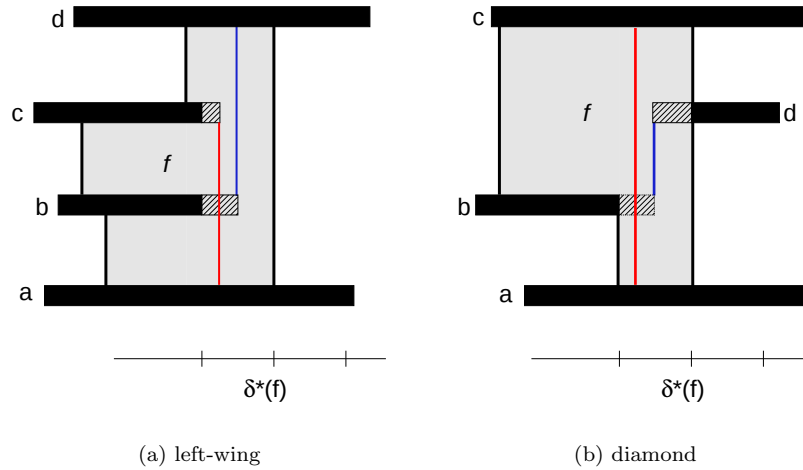


Figure 5: Re-inserting a pair of crossing edges  $(a, c)$  and  $(b, d)$  in face  $f$  by an extension of the inner vertex-segments, where  $(a, c)$  crosses  $b$ .

crossing in  $f$ . The vertex-segments of the extreme vertices cover the range from  $\delta^*(f) - 1$  to  $\delta^*(f)$ , and generally go far beyond.  $\square$

Finally, consider a separation pair  $\{u, v\}$  and its 3-connected components  $C_0, \dots, C_{k-1}$ , which are separated by separation edges  $e_1, \dots, e_k$  as copies of  $e_0 = (u, v)$ . Associate  $e_i$  with  $C_i$ . Then the 3-connected components are sandwiched between the vertex-segments of  $u$  and  $v$  and two adjacent components  $C_{i-1}$  and  $C_i$  are clearly separated by  $e_i$  in a left-to-right order, which is due to the  $st$ - and  $s^*t^*$ -numberings.

We can now establish our main result.

**Theorem 1** *There is a linear time algorithm to construct a 1-visibility representation of an embedded 1-planar graph on a grid of size at most  $(8n - 20) \times (n - 1)$ .*

**Proof:** First consider the case that the graph  $G$  is 3-connected. Its embedding is transformed into normal form with all crossings as augmented X-configurations with the exception of at most one crossing in the outer face. Now each crossing of a pair of edges has its own face in the embedded planar graph which remains after the removal of all pairs of crossing edges. A crossing in the outer face is assigned to the face to the left of the inserted separation edge. Each such face is a quadrangle. This property also holds for 2-connected graphs by the separation edges between 3-connected components. Hence, the planar graph after the extraction of all pairs of crossing edges can be drawn by PLANAR-VISIBILITY, and the extracted pairs edges can be re-inserted by CROSSING-INSERTION. This induces the crossing of a single vertex-edge pair for each pair of crossing edges in  $f$ , as shown in Lemma 4, such that each edge is crossed at most once.

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**Algorithm 3: 1-VISIBILITY**


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**Input:** A 1-planar embedding  $\mathcal{E}(G)$  of a 2-connected 1-planar graph  $G$ .

**Output:** A 1-visibility representation  $\Gamma$  on a grid of quadratic size.

- 1 Augment  $\mathcal{E}(G)$  to a planar maximal 1-planar embedding and update  $G$
  - 2 Decompose  $G$  into its 3-connected components.
  - 3 **foreach** *separating pair*  $\{u, v\}$  **do**
  - 4      $\left[ \begin{array}{l} \text{In } \mathcal{E}(G), \text{ add a copy of } (u, v) \text{ as a separating edge to the right of each} \\ \text{3-connected component at } u \text{ and update } G. \end{array} \right.$
  - 5 If edges  $(a, b)$  and  $(c, d)$  cross in the outer face of  $\mathcal{E}(G)$  with  $a, b$  in the outer face, then add a copy of  $(a, b)$  to  $G$  and  $\mathcal{E}(G)$  such that  $(a, b)$  and its copy are in the outer face.
  - 6 Transform  $\mathcal{E}(G)$  into normal form by transforming the embedding of each 3-connected component.
  - 7 Remove pairs of crossing edges from  $\mathcal{E}(G)$  via the removal of the crossing points in the planarization of  $\mathcal{E}(G)$ . Let  $\mathcal{E}(G_p)$  be the remaining planar embedding of the spanning planar subgraph  $G_p$  of  $G$ . Assign each pair of crossing edges to the face of  $\mathcal{E}(G_p)$  from which it was extracted.
  - 8 Construct a planar visibility representation  $\Gamma$  of  $G_p$  by PLANAR-VISIBILITY.
  - 9 (Separately for each 3-connected component) Compute the set of crossed vertex-segments by a maximum bipartite matching on the pairs of crossing edges and the non top and non bottom vertices of the faces from which they were extracted.
  - 10 Re-insert each pair of crossing edges into the face from which it was extracted using CROSSING-INSERTION .
  - 11 Scale all x-coordinates of  $\Gamma$  by the factor 4.
  - 12 Remove the edges from  $\Gamma$  that were added in Steps 1,3 and 4.
  - 13 Return  $\Gamma$ .
-

Multiple vertex crossings are excluded by a maximum matching between the set of faces  $F$  with a crossing and the set of inner vertices  $I$  associated with the faces of  $F$ . An inner vertex of a face is not the top or bottom vertex. By the  $st$ -numbering each vertex  $v$  is an inner vertex of at most two faces, one to the left and one to the right.  $v$  can be the top or bottom vertex of other faces. Hence,  $v$  is assigned to at most two faces of  $F$ , and each  $f \in F$  has two inner vertices, as can be seen from the left-wing, right-wing or diamond shape. The maximum bipartite matching problem over  $F$  and  $I$  has a solution by Hall's marriage theorem [18], since for every subset  $F' \subseteq F$  the number of inner vertices  $|I'|$  of the faces from  $F'$  is greater or equal to  $|F'|$ .

In this particular case, a maximum matching can be computed in linear time by first matching all inner vertices of degree one, and then matching the remaining faces using at most one alternation. Since the remaining faces and inner vertices all have degree two, the bipartite graph decomposes into disjoint alternating cycles.

PLANAR-VISIBILITY computes grid points for the segments and uses an area of at most  $(2n - 5) \times (n - 1)$  including the separation edges. The number of faces of the augmented graph  $G''$  is bounded from above by  $2n - 4$ , since for each separation edge there is a missing edge between the adjacent 3-connected components. CROSSING-INSERTION does not increase the area, but needs a scaling of the  $x$ -coordinates by four, which results in an area of at most  $(8n - 20) \times (n - 1)$ .

All steps take linear time. In 1-VISIBILITY steps 1-4 and 7 are done on planar graphs (including crossing points). The linear running time of step 6 is from [2] and of step 8 from [10, 26, 32]. Step 10 takes  $\mathcal{O}(1)$  time per crossing, and there are at most  $n - 2$  crossings, step 5 is a single action, and steps 11 and 12 take  $\mathcal{O}(1)$  time per item and thus  $\mathcal{O}(1)$  time in total. Finally, the linear time bound of step 9 is shown above.  $\square$

**Corollary 1** *Every 1-planar graph is a 1-visible graph, and thus a bar 1-visibility graph.*

## 4 Density

It is easily seen that 1-visible graphs of size  $n$  have at most  $4n - 8$  edges, since there are at most  $3n - 6$  planar edges and at most  $n - 2$  edges whose segments cross a vertex-segment. This is exactly the upper bound of the density of 1-planar graphs.

**Lemma 5** *A 1-visible graph of size  $n$  has at most  $4n - 8$  edges.*

From Corollary 1 we obtain a new and simple proof for the maximal density of 1-planar graphs, which was proved before in [4, 15, 23].

**Corollary 2** *A 1-planar graph of size  $n$  has at most  $4n - 8$  edges.*

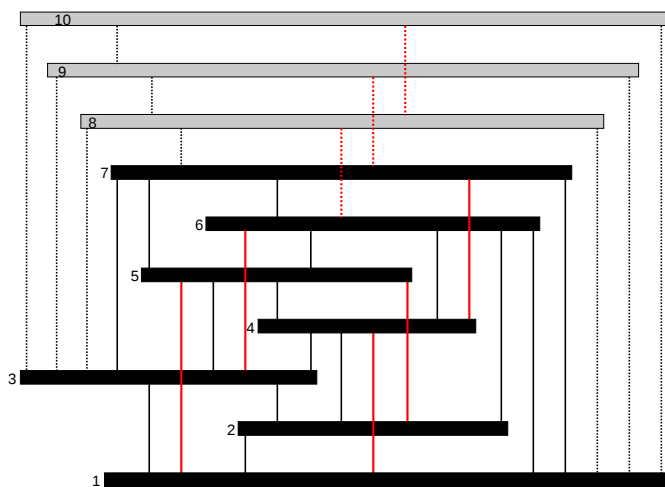


Figure 6: The  $K_7-e$  graph with the vertices  $\{1, \dots, 7\}$  is 1-visible and not 1-planar. The edge  $(2, 7)$  is missing. The graph can be expanded by new vertices which each add four edges.

However, there are 1-visible graphs which are not 1-planar, even if they have the maximum of  $4n - 8$  edges.

**Theorem 2** *For every  $n \geq 7$  there are graphs with  $4n - 8$  edges which are 1-visible and not 1-planar.*

**Proof:** There are no 1-planar graphs with  $n = 7$  (or  $n = 9$ ) vertices and  $4n - 8$  edges [5, 31], however, the complete graph on 7 vertices without one edge  $K_7-e$  is 1-visible, as shown in Fig. 6.

For  $n \geq 8$  construct the graph  $G_n$  from  $K_7-e$  and add  $n - 7$  vertices and connect each such  $v_i$  with vertex 3 on the left and with vertex 1 on the right side and with  $v_{i-1}$  and  $v_{i-2}$  on top, where the edge  $(v_i, v_{i-2})$  crosses  $v_{i-1}$ , as illustrated in Fig. 6.

Since the 1-planar graphs have the subgraph property  $G_n$  is not 1-planar.  $\square$

1-planar (1-visible) graphs with  $4n - 8$  edges are called *optimal* [5, 31]. Note that there are optimal 1-planar graphs only for  $n = 8$  and  $n \geq 10$  [5, 31], whereas there are optimal 1-visible graphs for every  $n \geq 7$ . More 1-visible and not 1-planar graphs can be constructed using the schema of Fig. 7, where the outer frame represents a subgraph with a unique 1-planar embedding as in [21] and the edge-segment  $(a, c)$  crosses vertex-segment  $b$  and would cross at least two edges in every 1-planar drawing.

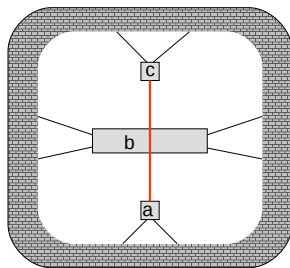


Figure 7: Schema for non 1-planar 1-visibility graphs.

## 5 Conclusion and Perspectives

We showed that single edge-vertex segment crossings in visibility representations are more powerful than single edge-edge crossings in common drawings. Visibility representations can be used to define further classes of beyond planar graphs, e.g., by single edge-vertex and edge-edge crossings in the flat visibility approach of Biedl [3] or in 2-dimensional visibility approaches [20]. We conjecture that the recognition problem for 1-visible graphs is NP-hard.

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