



Vertex Intersection Graphs of Paths on a Grid

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Abstract

We investigate the class of vertex intersection graphs of paths on a grid, and specifically consider the subclasses that are obtained when each path in the representation has at most k bends (turns). We call such a subclass the B_k -VPG graphs, $k \geq 0$. In chip manufacturing, circuit layout is modeled as paths (wires) on a grid, where it is natural to constrain the number of bends per wire for reasons of feasibility and to reduce the cost of the chip.

If the number k of bends is not restricted, then the VPG graphs are equivalent to the well-known class of string graphs, namely, the intersection graphs of arbitrary curves in the plane. In the case of B_0 -VPG graphs, we observe that horizontal and vertical segments have strong Helly number 2, and thus the clique problem has polynomial-time complexity, given the path representation. The recognition and coloring problems for B_0 -VPG graphs, however, are NP-complete. We give a 2-approximation algorithm for coloring B_0 -VPG graphs. Furthermore, we prove that triangle-free B_0 -VPG graphs are 4-colorable, and this is best possible.

We present a hierarchy of VPG graphs relating them to other known families of graphs. The grid intersection graphs are shown to be equivalent to the bipartite B_0 -VPG graphs and the circle graphs are strictly contained in B_1 -VPG. We prove the strict containment of B_0 -VPG into B_1 -VPG, and we conjecture that, in general, this strict containment continues for all values of k . We present a graph which is not in B_1 -VPG. Planar graphs are known to be in the class of string graphs, and we prove here that planar graphs are B_3 -VPG graphs, although it is not known if this is best possible.

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1 Introduction

Let \mathcal{P} be a set of simple paths on a (rectangular) grid. The vertex intersection graph $\text{VPG}(\mathcal{P})$ of \mathcal{P} has vertex set V , where every vertex $v \in V$ corresponds to a path $P_v \in \mathcal{P}$, and edge set E , where $(u, v) \in E$ if and only if the corresponding paths P_u and P_v intersect, i.e., $E = \{(u, v) \mid u, v \in V, u \neq v, P_u \cap P_v \neq \emptyset\}$. We call a graph G a *VPG graph* if $G = \text{VPG}(\mathcal{P})$, for some \mathcal{P} . In this paper, if not specified otherwise, two paths “intersect” by sharing a vertex of the grid (grid point). If \mathcal{P} is a set of simple paths on a grid, where each path has at most k bends (90° turns), then the graph G is called B_k -VPG.

B_k -VPG graphs are related to several other families of intersection graphs that have been studied in the literature. String graphs are the intersection graphs of curves (i.e., arbitrary paths) in the plane, defined in a similar way to the VPG graphs: here the vertex set corresponds to a set of curves in the plane and two vertices are adjacent if and only if the corresponding curves intersect. It is rather simple to prove that the string graphs are equivalent to the VPG graphs when there is no restriction on the number of bends per path in the grid, see Theorem 1.

Our focus here is to constrain the number of bends, studying the properties of B_k -VPG graphs for particular values of k or for bounded values of k . Interval graphs and trees are both subfamilies of B_0 -VPG, and the so called grid intersection graphs of Hartman, et al. [18] are equivalent to the bipartite B_0 -VPG graphs, see Theorem 2.

Circle graphs are a subfamily of string graphs, and it is easy to show that they are contained in the class B_1 -VPG (Theorem 3). This immediately implies that the coloring problem is NP-complete on B_1 -VPG graphs. We prove the stronger result that the coloring problem is NP-complete for B_0 -VPG graphs (Theorem 8).

For planar graphs, Sinden [32] showed that every planar graph is a string graph. In this paper, we show that every planar graph is a B_3 -VPG graph (Theorem 7), and we conjecture that this is best possible.

Another connection between planar graphs and string graphs began when Scheinerman [31] conjectured that planar graphs are contained in the family of segment graphs (SEG), the intersection graphs of straight-line segments in the plane with an arbitrary number of directions. Recently, Chalopin and Gonçalves [6] proved Scheinerman’s conjecture. West [33] proposed a stronger version of Scheinerman’s conjecture, namely, that every planar graph has a SEG representation where the segments have four possible directions. Previously, Hartman et al. [18] and de Fraysseix et al. [12] had proven the conjecture for bipartite planar graphs using only two possible directions for the segments. Castro et al. [11] proved the conjecture for triangle-free planar graphs, allowing only three directions for the segments. Chalopin et al. [7] proved that planar graphs are in 1-String, which are the family of string graphs where two curves are allowed to intersect only once.

The paper is organized as follows. In Section 2, we first present relationships between B_k -VPG graphs and four other known families of graphs. In Section 3,

we show that any collection of horizontal and vertical segments has strong Helly number 2. We then study the structure of chordless cycles in B_0 -VPG graphs, and prove that B_0 -VPG graphs are sun-free. This property implies that chordal B_0 -VPG graphs are equivalent to strongly chordal B_0 -VPG graphs. The proof that there is a string graph which is not in B_1 -VPG is given in Section 4. In Section 5, we present a complete hierarchy of B_k -VPG graphs, $k \geq 0$, and other known families of graphs. Planar graphs are discussed in Section 6 where we prove that every planar graph is a B_3 -VPG graph. The minimum coloring problem on B_0 -VPG is proved to be NP-complete in Section 7, and we show a 2-approximation algorithm for coloring B_0 -VPG graphs. In Section 8, we prove that triangle-free B_0 -VPG graphs are 4-colorable, and that this bound is tight. Finally, some open problems are given in Section 9.

Motivation.

The motivation for studying B_k -VPG graphs is as follows. String graph problems arose naturally in the context of layout problem of integrated thin film RC circuits [32], since the technology for creating the circuits made it possible for some pairs of conductors to cross. Later on, the circuit layout setting was modeled as paths (wires) intersecting on a grid. In the knock-knee layout models [1, 27], each wire may bend (turn) at a grid point but is not allowed to intersect with another wire. A layout may have multiple layers and in a legal layout, on each layer, the vertex intersection graph of paths on a grid is an independent set. This corresponds to a graph coloring problem. We adopt this model to investigate the properties of VPG graphs. The minimum coloring problem of VPG graphs defines the knock-knee multiple layout with minimum number of layers.

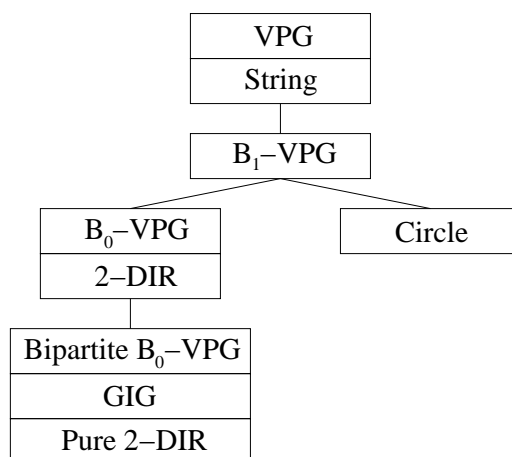
In chip manufacturing, each wire bend requires a transition hole. A large number of such holes may increase the layout area and the cost of the chip. Much research has been done to minimize the number of bends in a layout and on layout optimization problems in general. We apply here the constraint on the number of bends per wire.

2 Preliminaries: The relationship of B_k -VPG graphs to four other classical families of graphs

We begin our study by stating some simple relationships between the B_k -VPG graphs and the well known families of string graphs, d -DIR graphs for $d=2$, grid intersection graphs (GIG) and circle graphs, as shown in Figure 1.

2.1 The equivalence of VPG and string graphs

The equivalence of the two families, string graphs and VPG graphs, has been a folklore in the graph theory community (for example, see [8]). In the following theorem, we state the equivalence explicitly.

Figure 1: Relations between B_k -VPG graphs and well known graph classes.

Theorem 1 *The family of VPG graphs is equivalent to the family of string graphs.*

Proof: Obviously, every path in the grid is a particular case of a curve on a plane. Therefore, every VPG graph is a string graph. In the other direction, consider a string representation. First, we may assume, without loss of generality, that at most two strings meet in any intersection point [21]. In such a representation, each intersection point has degree at most 4. Second, a sufficiently fine grid can be imposed upon the plane to allow each curve to be embedded as a path on the grid, by replacing segments of the curve by “staircases” of vertical and horizontal lines with sufficiently small distance ϵ between them, such that (1) all intersection points of the curves fall on grid points and (2) no new intersections are introduced. This transforms the string representation into a VPG representation without creating new adjacencies in the graph, showing that every string graph is a VPG graph. \square

Remark 2.1 *Consider the complete graph of size five and subdivide every edge into two edges by adding a new vertex. In [32], it was shown that this graph, denoted here by C_5^2 , is not a string graph, see Figure 5.*

2.2 The equivalence of bipartite B_0 -VPG and GIG graphs

The Grid Intersection Graphs (GIG) are the intersection graphs of horizontal and vertical line segments on the plane, such that no two horizontal segments or two vertical segments intersect [18]¹.

¹More precisely, they allow intersecting intervals on the same horizontal or vertical line, but do not count this as producing an edge in the graph. Hence, without loss of generality, one may assume that, by perturbing such intervals by a small epsilon, they do not intersect.

Theorem 2 *The family of bipartite B_0 -VPG graphs is equivalent to the family of GIG.*

Proof: Obviously, every GIG graph is bipartite and B_0 -VPG. Conversely, a B_0 -VPG graph can be represented as intersection of rectangles on the plane, by taking a B_0 -VPG representation on the grid and thickening the segments (paths). The intersection graphs of rectangles on the plane are also known as graphs with boxicity two.

In [2], it was proved that the grid intersection graphs (GIG) are the bipartite graphs with boxicity two. This implies that every bipartite B_0 -VPG graph is a GIG. \square

Remark 2.2 *It is easy to show that every tree is a bipartite B_0 -VPG graph.*

2.3 The equivalence of B_0 -VPG and 2-DIR graphs

Two other subfamilies of string graphs are of interest. The d -DIR graphs are the family of intersection graphs of straight-line segments parallel with at most d directions. The PURE- d -DIR graphs have a representation by straight-line segments parallel with at most d directions, such that every two parallel segments are disjoint [23]. In the case of $d = 2$, without loss of generality, it is possible to assume that the two directions are orthogonal. Thus, we observe the following.

Remark 2.3 *The B_0 -VPG graphs are exactly the 2-DIR graphs; the PURE-2-DIR graphs are exactly the GIG graphs.*

2.4 The containment of circle graphs in B_1 -VPG

Circle graphs are the intersection graphs of chords in a circle. The graph W_n , the *wheel* graph, consists of a chordless cycle of size n ($n \geq 4$) and a universal vertex.

Theorem 3 *The family of circle graphs is strictly contained in the family of B_1 -VPG graphs; the wheel W_5 is a separating example.*

Proof: Consider a representation of G as intersecting chords of a circle. We slide all the endpoints of the chords to the upper left quarter of the circle, while preserving their order (thus, intersections are not changed.) Now, replace each chord by a path with one bend, see Figure 2. All the adjacencies of the graph remain unchanged in the resulting B_1 -VPG representation. Thus, every circle graph is a B_1 -VPG graph.

The wheel W_4 is a circle graph, but none of the wheels W_n ($n \geq 5$) is a circle graph, as shown in [5]. All the wheels are easily seen to be B_1 -VPG graphs. Therefore, the containment is strict. \square

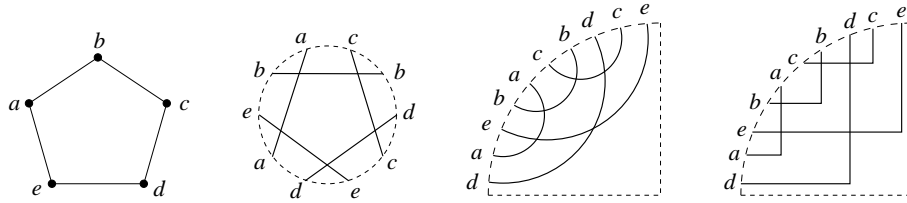


Figure 2: The transformation of a circle graph representation into a B_1 -VPG representation.

3 Structural results on B_0 -VPG graphs

3.1 Helly properties of B_0 -VPG graphs

Let $\mathcal{S} = \{S_i\}_{i \in I}$ be a collection of subsets of a set S . We say that \mathcal{S} has *Helly number* h if, for all $J \subseteq I$, $\bigcap \{S_i | i \in J\} = \emptyset$ implies that there exist h indices $i_1, \dots, i_h \in J$ such that $S_{i_1} \cap \dots \cap S_{i_h} = \emptyset$. We say that \mathcal{S} has *strong Helly number* s if, for all $J \subseteq I$, there exist s indices $i_1, \dots, i_s \in J$ such that $S_{i_1} \cap \dots \cap S_{i_s} = \bigcap \{S_i | i \in J\}$.

It is well-known [3] that any collection of paths on a tree has Helly number 2, and any collection of intervals on a line has strong Helly number 2. We see here that orthogonal segments also have strong Helly number 2.

Proposition 3.1 *Any collection of horizontal and vertical segments on a grid has strong Helly number 2.*

Proof: Consider a set of horizontal and vertical segments $\{P_i | i \in J\}$. If there is a pair of segments P_a, P_b that do not intersect, then $P_a \cap P_b = \emptyset = \bigcap \{P_i | i \in J\}$, and we are done.

So we may assume that every two segments intersect. If all the segments are horizontal on the grid, then they are intervals on the same grid line, and therefore clearly they have strong Helly number 2. The same is true for only vertical paths.

Otherwise, there is a horizontal segment P_a and a vertical segment P_b , and they intersect in one common grid point $P_a \cap P_b = q$. Clearly, the intersection of each horizontal segment with P_b is the point q , and the intersection of each vertical segment with P_a is the point q , so $\bigcap \{P_i | i \in J\} = q = P_a \cap P_b$. \square

Corollary 3.2 *In a B_0 -VPG representation of a clique, all the corresponding paths share a common grid point.*

Example 3.3 *Let G be a graph that contains a clique K_m ($m \geq 5$) such that 5 or more vertices of the clique have private neighbors. Then the graph is not B_0 -VPG.*

Remark 3.4 A set of single bend paths on a grid does not necessarily have Helly number 3; for an example, consider the four paths $(0,0)-(0,1)-(1,1)$, $(0,1)-(1,1)-(1,0)$, $(1,1)-(1,0)-(0,0)$, $(1,0)-(0,0)-(0,1)$. The common intersection of these four paths is empty, yet the intersection of any three is non-empty.

Golumbic, Lipshteyn and Stern [17] have shown that any collection of single bend paths on a grid has strong Helly number 4.

3.2 Suns are not in B_0 -VPG

The graph S_n (n -sun) consists of a clique $K=\{v_1, \dots, v_n\}$ of size n and an independent set $S=\{v_{n+1}, \dots, v_{2n}\}$ of size n , where vertex v_{i+n} is adjacent to v_i and v_{i+1} (modulo n), for $i = 1, \dots, n$. The *strongly chordal graphs* can be defined as the sun-free chordal graphs [15].

Theorem 4 A B_0 -VPG graph contains no induced S_n for $n \geq 3$. Therefore, the family of chordal B_0 -VPG graphs is equivalent to the family of strongly chordal B_0 -VPG graphs.

Proof: Suppose S_n is a B_0 -VPG graph for some $n \geq 3$. Let \mathcal{P}_K and \mathcal{P}_S be the sets of paths that correspond to the vertex sets K and S , respectively. By Corollary 3.2, all paths in \mathcal{P}_K share a common grid point q . If all paths in \mathcal{P}_K are on the same line, say horizontal, then clearly only at most two vertices in S can be represented. Therefore, there is a horizontal path P_i and a vertical path P_j in \mathcal{P}_K and $j = i + 1 \pmod n$. To represent a sun, there must exist a path in \mathcal{P}_S that intersects with both P_i and P_j and hence contains the common grid point q of \mathcal{P}_K , a contradiction. \square

Remark 3.5 It can be easily shown that the suns S_n are in B_1 -VPG for all $n \geq 3$. Figure 3 illustrates the case for $n = 4$ and the general case.

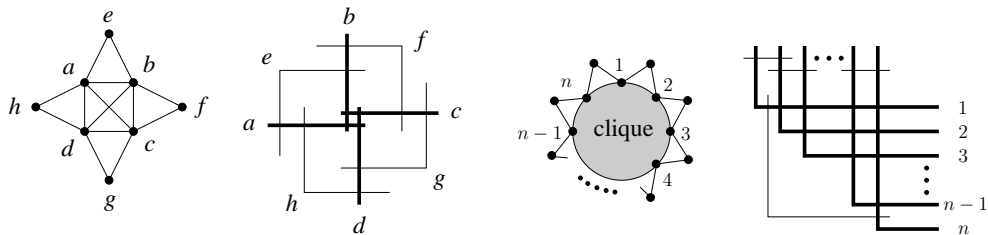


Figure 3: The graphs S_4 and in general $S_n(n \geq 3)$ each with a B_1 -VPG representation. The bold paths correspond to the central clique.

3.3 Cycles in B_0 -VPG graphs

In general, chordless cycles have many possible B_0 -VPG representations, however it is easy to see that the chordless 4-cycle has only one.

Remark 3.6 *The graph C_4 has a unique B_0 -VPG representation; it consists of two horizontal parallel paths intersecting with two vertical parallel paths. The uniqueness is up to isomorphism that is, the four intersections points of the four paths close a rectangle on the hosting grid.*

An example of this useful remark is the following.

Proposition 3.7 *Let G be a connected graph in which each edge belongs to an induced C_4 . If G is a B_0 -VPG graph, then it is bipartite, hence also a GIG.*

Proof: If G is not bipartite, it has an odd cycle. Thus, there are two vertices x, y connected by an edge, whose paths have the same direction (both horizontal or both vertical). However, the edge xy must belong to an induced C_4 , and thus by Remark 3.6, x and y must have different directions, a contradiction. Therefore, G is bipartite. By Theorem 2, G is a GIG. \square

4 Going beyond B_1 : the graph K_n^3 ($n \geq 33$)

In this section, we prove that there is a string graph which is not a B_1 -VPG graph. We note that a method for constructing string graphs, that require arbitrary number of bends, is presented in [24].

Let K_n^3 be the split graph which consists of a clique K of n vertices labeled by $[n] = \{1, 2, \dots, n\}$ and an independent set I of $\binom{n}{3}$ vertices which are labeled by subsets of $[n]$ of size 3. The edges between K and I are defined by membership, for example, only the vertices 2, 5, and 8 of K are connected to the vertex $\{2, 5, 8\}$ of I . Since K_n^3 is a chordal graph, it is a string graph [21].

Theorem 5 *The graph K_{33}^3 is not B_1 -VPG.*

Proof: Suppose there exists a B_1 -VPG representation of K_{33}^3 . For each vertex v , denote the corresponding path by P_v (we write P_{123} rather than $P_{\{1,2,3\}}$, etc.)

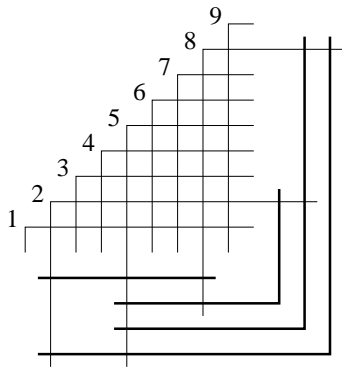
Since the vertices of K are represented by paths with one bend², each of them is either \sqcap , \sqsupset , \sqcup or \sqsubset . Without loss of generality, at least $9 = \lceil \frac{1}{4} \cdot 33 \rceil$ of them are \sqcap . Also without loss of generality, these are the vertices labeled by $1, 2, \dots, 9$. Since they form a clique, the bending points of the corresponding paths P_1, P_2, \dots, P_9 form a weakly ascending sequence, i.e., for $i < j$, the bending points (x_i, y_i) and (x_j, y_j) of P_i and P_j , respectively, we have $x_i \leq x_j$ and $y_i \leq y_j$. Moreover, using standard arguments of small changes in a construction, we may assume that the paths P_1, P_2, \dots, P_9 are in fact strictly ascending, i.e., $x_i < x_j$ and $y_i < y_j$, for $i < j$.

Without loss of generality, there exists a permutation π of $\{1, \dots, 9\}$, with $P_{\pi(1)}, \dots, P_{\pi(9)}$ forming a weakly ascending sequence.

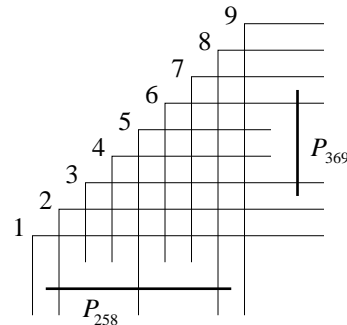
Consider the paths P_{147}, P_{258} and P_{369} .

²Without loss of generality, for any path with no bend we can add an ϵ -length orthogonal segment at one of its end points without introducing new adjacencies.

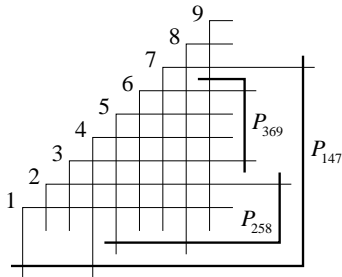
Assume, without loss of generality, that P_{258} meets the vertical part of P_5 . There are four ways in which P_{258} can meet P_2 and P_8 (P_{258} can meet a vertical or a horizontal part of each of them); they are shown by bold lines in Figure 4(a).



(a) Bold lines denote all possible ways to add P_{258} .



(b) The path P_{258} and part of the path P_{369} .



(c) The only possible way to insert P_{147} , P_{258} and P_{369} .

Figure 4: Illustrations for proof of Theorem 5.

However, in most of these cases it is impossible to add P_{147} and P_{369} to the construction. Consider, for example, the case when P_{258} meets the vertical parts of P_2 , P_5 and P_8 , as shown in Figure 4(b).

Notice that in the discussed case, P_{369} necessarily meets *horizontal* parts of P_3 and of P_6 (a corresponding part of P_{369} is shown in Figure 4(b)) since otherwise it cannot meet all of P_3 , P_6 and P_9 . However, then P_{147} cannot meet all of P_1 , P_4 and P_7 .

By carefully examining of all possible cases, we find that the only way to add P_{147} , P_{258} and P_{369} is as shown in Figure 4(c).

However, now it is impossible to add the path P_{169} . □

Corollary 4.1 *The graph K_n^3 is not B_1 -VPG, for $n \geq 33$.*

Remark 4.2 *Dangelmayr, Felsner and Trotter [10] have recently shown that the graph K_n^3 , for $n \geq 33$, is not an intersection graph of curves on a plane,*

where every pair of curves intersect in at most one point. Our result does not follow from theirs, since in our case paths can intersect in two points, or along a segment.

5 The complete hierarchy of B_k -VPG graphs

In this section, we demonstrate the results illustrated in the complete hierarchy shown in Figure 5. We say that a hierarchy is *complete*, when all containment relationships are given. That is, (1) classes that appear in the same box are equivalent, (2) a downward edge from class A to class B indicates that class A contains class B, (3) an example appearing along the edge between two classes is a separating example for those classes, (4) the lack of a hierarchical (containment) relation indicates that the classes are incomparable.

Theorem 6 *The hierarchy of relationships shown in Figure 5 is complete.*

Proof:

1. Equivalences

All the equivalence relations are summarized in Table 1.

2. Containments

We have already shown in Theorem 3 that the circle graphs are B_1 -VPG. By [21], all chordal graphs are string graphs. Having already shown the equivalence of classes in the same box, all the other containment relations in the hierarchy are trivial by their definitions.

3. Separating examples

The following are the separating examples as shown in Figure 5.

The graph S_4 : As proved in Theorem 4, the suns are not B_0 -VPG graphs. They are well known to be chordal and have a simple circle representation. By Remark 3.5, they are B_1 -VPG graphs.

The graph BW_3 : By Proposition 3.7, the graph BW_3 is GIG. However, it is not a circle graph.

Result	Where
string graphs \equiv VPG	Theorem 1
2-DIR \equiv B_0 -VPG	Remark 2.3
PURE-2-DIR \equiv GIG \equiv bipartite B_0 -VPG	Remark 2.3 and Theorem 2
chordal B_0 -VPG \equiv strongly chordal B_0 -VPG	Theorem 4

Table 1: Justification for the equivalences in the hierarchy.

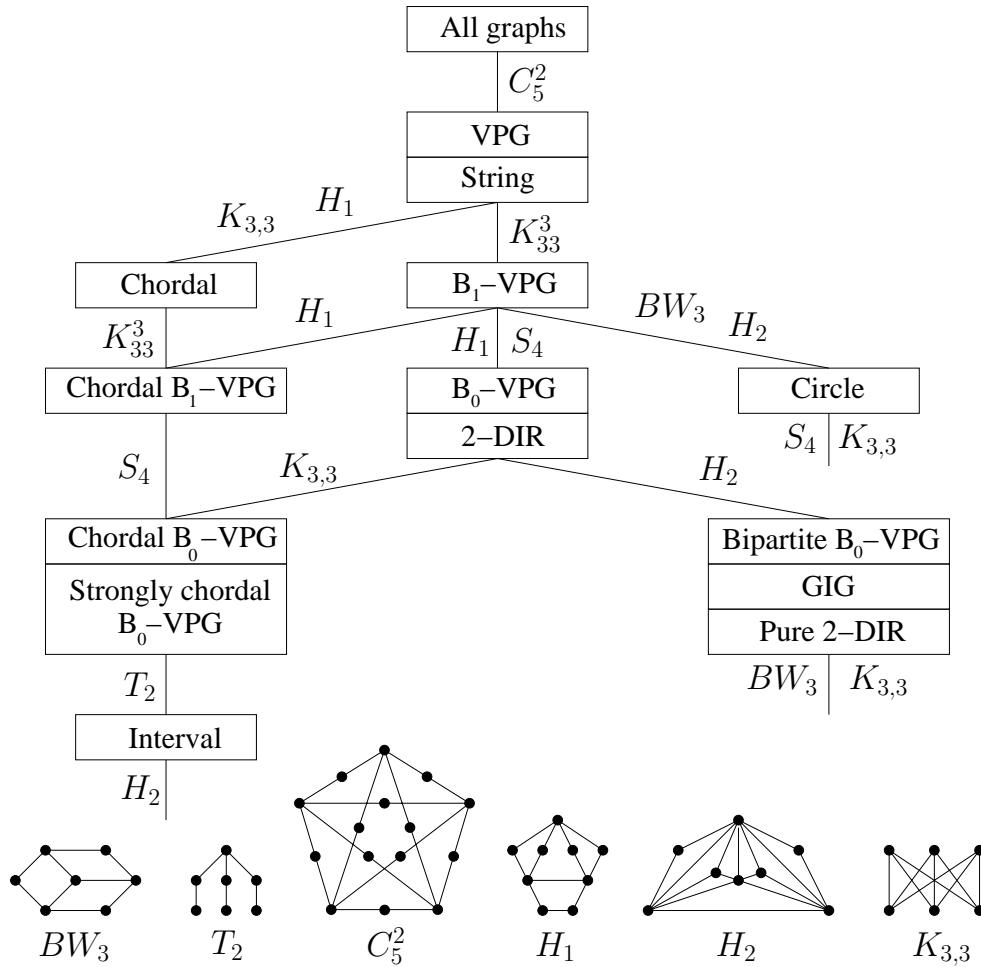


Figure 5: Hierarchy of B_k -VPG graphs and other graphs.

The graph $K_{3,3}$: The graph $K_{3,3}$ obviously has trivial GIG and circle representations. However, it is not chordal since it contains a C_4 .

The graph H_2 : In [9], the graph H_2 is proved to be an interval graph but not a circle graph.

The graph $K_{3,3}^3$: By Corollary 4.1, the graph $K_{3,3}^3$ is a string graph which is not B_1 -VPG. It is well known to be a chordal graph.

The graph C_5^2 : In [32] it is shown that the graph C_5^2 is not a string graph.

The graph H_1 : By Proposition 3.7, the graph H_1 is not a B_0 -VPG.

Figure 6 shows a B_1 -VPG representation for H_1 .

The graph T_2 : It is well known that the graph T_2 is not an interval graph.

4. Incomparabilities.

The separating examples between incomparable classes of graphs are given in Table 2.

X	Y	$G_1 \in X - Y$	$G_2 \in Y - X$
Circle	B_0 -VPG, GIG	S_4	BW_3
Circle, GIG	Chordal, Chordal B_1 -VPG Chordal B_0 -VPG, Interval	$K_{3,3}$	H_2
B_1 -VPG	Chordal	$K_{3,3}$	K_{36}^3
B_0 -VPG	Chordal, Chordal B_1 -VPG	$K_{3,3}$	S_4

Table 2: Separating examples between incomparable classes of graphs in the hierarchy.

We have shown all the containment relationships in the hierarchy, and therefore the hierarchy is complete. \square

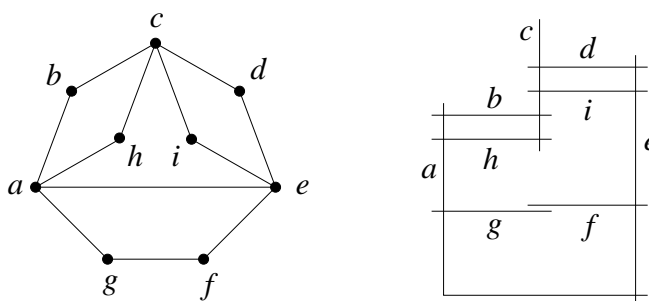


Figure 6: The balloon graph H_1 and its B_1 -VPG representation.

6 Planarity and B_k -VPG Graphs

As mentioned in Section 1, Sinden [32] showed that every planar graph is a string graph. Since the VPG graphs are equivalent to the string graphs (Theorem 1), we ask the question, what is the smallest number of bends needed to represent a planar graph? In this section, we show that every planar graph is a B_3 -VPG graph.

Theorem 7 *Every planar graph is a B_3 -VPG graph.*

Proof: Let G be a planar graph. By a result of de Fraysseix, Ossona de Mendez and Rosenstiehl [13], we can represent G with a “ τ -contact system”, where to each vertex of G there corresponds a τ -shape, such that two τ -shapes may touch but may not cross, and there is an edge between vertices in G if their corresponding τ -shapes touch. Without loss of generality, we may assume that in such a τ -representation, all horizontal segments of the τ s belong to different lines.

In similar fashion to [4], (where it was proven that every planar graph is B_5 -EPG), we transform each τ -shape into a path with 3 bends as indicated in Figure 7. By that, we obtain a B_3 -VPG representation of G . \square

Conjecture 6.1 *We suspect that the planar graphs are not contained in B_2 -VPG, i.e., that there exists a planar graph which is not B_2 -VPG.*

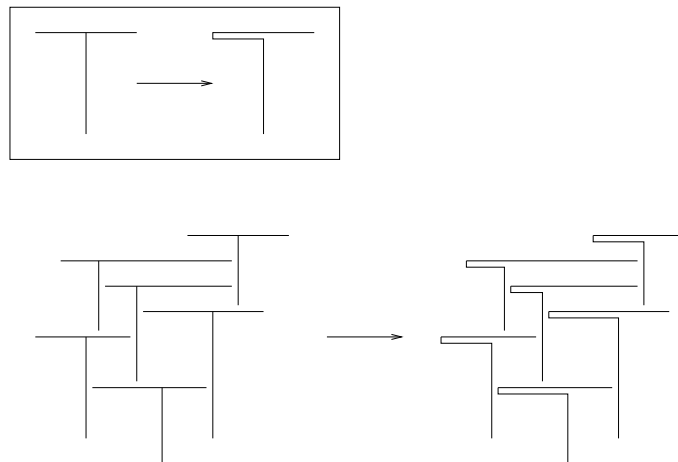


Figure 7: Transformation of τ -shapes representation into a B_3 -VPG representation.

7 Complexity results for B_k -VPG graphs

7.1 Generally known results

The recognition of string graphs was first shown to be NP-hard by [22]. In [25], an example of a string graph with exponential number of intersection points was given, posing the question whether recognizing string graphs is decidable. After a decade, it was proved affirmatively (independently) in [28, 30]. Surprisingly, by [29], it was proven to be in NP. Hence, the recognition of string graphs is NP-complete. In [23], the recognition problems of d -DIR and PURE- d -DIR were proven to be NP-complete, for $d \geq 2$.

Putting these all together, since (a) VPG and string graphs are equivalent (Theorem 1), (b) B_0 -VPG graphs and 2-DIR graphs are equivalent (Remark 2.3), and (c) bipartite B_0 -VPG graphs, GIG and PURE-2-DIR graphs are equivalent (Theorem 2 and Remark 2.3), we obtain:

Corollary 7.1 *The recognition problems for VPG graphs, B_0 -VPG graphs and bipartite B_0 -VPG graphs are all NP-complete.*

It is therefore likely that recognizing B_k -VPG graphs for any $k > 0$ is also NP-complete, but we leave this as an open problem.

Using another result of [20] for the NP-completeness of the maximum independent set problem on 2-DIR graphs, and similarly, by combining the results of [18] and [19] for the Hamiltonian graph problems on the class GIG, we obtain:

Corollary 7.2 *The maximum independent set problem on B_0 -VPG graphs is NP-complete. The Hamiltonian path problem and the Hamiltonian circuit problem on B_0 -VPG graphs are NP-complete.*

It is shown in [26] that the problem of maximum clique for B_1 -VPG graphs is NP-complete. Hence, finding maximum clique in B_k -VPG graphs is NP-complete, for $k \geq 1$. In contrast to this, for B_0 -VPG graphs, given a B_0 -VPG representation of a graph G the maximum clique problem can be solved very efficiently: By Corollary 3.2, all corresponding paths of a clique have a common point on the grid, either at a “crosspoint” where horizontal segments meet vertical segments, or at an “internal” point involving only horizontal segments or only vertical segments. In a manner similar to finding the cliques of an interval graph representation, it is sufficient to traverse each horizontal, noting each segment endpoint and each crosspoint, then similarly traverse each vertical, to find the one with the maximum number of paths passing through it, namely, the collection of all paths passing through this grid point. This set of vertices form a maximum clique. This method has $O(n + m)$ time complexity where n is the number of vertices in the graph (segments) and m is the number of crosspoints and internal points on the grid. A polynomial algorithm for maximum clique for a generalization of B_0 -VPG graphs is presented in [20].

7.2 The complexity of coloring B_0 -VPG graphs

As we discussed in Section 1, coloring the paths in a VPG representation of a graph G , where two paths on the grid are colored with different colors if the paths share at least one grid point, has an important application in circuit layout problems. This motivates our study of coloring B_k -VPG graphs.

The minimum coloring of string graphs is NP-complete [14], so the same is true for VPG graphs in general, since they are equivalent to string graphs. This leads to the question of the complexity when restricting the number of bends per path in a VPG representation.

We showed that the family of circle graphs is contained in B_1 -VPG (Theorem 3), and we know that the problem of coloring a circle graph is NP-complete

[16]. Hence, it immediately follows that the coloring problem for B_k -VPG graphs is NP-complete for all $k \geq 1$. We will show the stronger result that the coloring problem for B_0 -VPG graphs is NP-complete. Let $\chi(G)$ be the chromatic number of G and let $\omega(G)$ be the clique number of G .

Theorem 8 *Let G be a B_0 -VPG graph and let m be a positive integer ($m \geq 3$). It is NP-complete to decide if $\chi(G) \leq m$.*

Proof: To prove the result, we first use the transformation of circle graphs to B_1 -VPG graphs presented in Theorem 3, namely, each chord can be replaced by a path with only one bend. We then associate to a circle graph G a graph G' , which is B_0 -VPG. We show that G is m -colorable if and only if G' is m -colorable.

The transformation is as follows: In a B_1 -VPG representation of G , each vertex v of G is represented by a path P_v . We replace this path by two disjoint segments denoted X_v and Y_v , one horizontal and one vertical. These two segments are not connected. To link X_v to Y_v , we add an $m - 1$ clique, at the bend point of the former path. We do this by adding $m - 1$ short paths touching both X_v and Y_v (this transformation is shown in Figure 8). We transform each path of the B_1 -VPG representation by this procedure. Note that each clique added at the bend point of each path is only connected to the horizontal and vertical paths that represent a vertex v . It is clear from the transformation that the obtained graph G' is a B_0 -VPG graph. Moreover, the transformation can be performed in polynomial time and the size of G' is polynomial in the size of G , since $|V(G')| = 2n + n \cdot (m - 1) \leq 2n + n^2$.

Now we prove that G is m -colorable if and only if G' is m -colorable. Let $\varphi : V \mapsto \{1, \dots, m\}$ be a valid assignment of colors for G . Then to color G' it suffices to color each vertex from G (i.e., the horizontal and vertical path) with the color used in G . Each clique is connected only to one vertical and one horizontal path, and by construction these two paths have the same color. We can use the $m - 1$ remaining colors to color the paths representing the clique.

We now show that if G' is m -colorable then G is m -colorable. The graph G' is m -colorable, therefore the $m - 1$ clique connecting any X_v and Y_v requires $m - 1$ colors. Consequently, X_v and Y_v have the same color. Since the coloring of G' is a valid assignment of colors, it is also a valid coloring of G : color the path representing v in G with the same color of the vertices in G' that correspond to X_v and Y_v . \square

Theorem 9 *For any B_0 -VPG graph G , $\omega(G) \leq \chi(G) \leq 2 \cdot \omega(G)$. Moreover, there is a polynomial 2-approximation scheme for coloring B_0 -VPG graphs.*

Proof: Consider the following greedy naive approximation algorithm for coloring B_0 -VPG graphs: Each horizontal or vertical line on the grid is an induced interval graph. We color all these interval subgraphs optimally, such that for horizontal lines we use c different colors and for vertical lines we use c' different colors. Note that the resulting coloring is not optimal. Thus, we get a polynomial 2-approximation scheme.

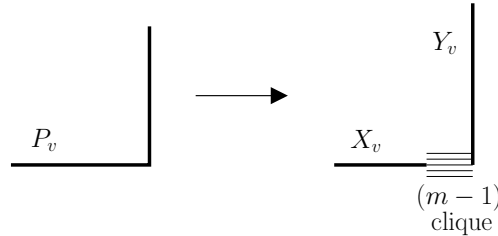


Figure 8: The transformation of a path with one bend to a B_0 -VPG representation.

We will show that $\max\{c, c'\} \leq \omega(G) \leq \chi(G) \leq c + c' \leq 2 \cdot \omega(G)$.

Since interval graphs are perfect graphs, c and c' correspond to the size of the maximum clique of horizontal lines and respectively of vertical lines. Therefore, $\max\{c, c'\} \leq \omega(G)$. For every graph, $\omega(G) \leq \chi(G)$, and clearly, $\chi(G) \leq c + c' \leq 2 \cdot \omega(G)$. This proves the theorem. \square

8 The chromatic number of triangle-free B_0 -VPG graphs

In this section, we construct a triangle-free B_0 -VPG graph with $\chi = 4$. Thus, the bound in Theorem 9 is tight for the case $\omega = 2$.

Theorem 10 *There is a triangle-free B_0 -VPG graph which is not 3-colorable.*

Proof: Step 1. Consider first the graph H whose B_0 -VPG representation is shown in Figure 9 (small black circles denote overlapping of paths). We claim that it is impossible to color H in three colors, using only two colors for horizontal paths.

To obtain a contradiction, suppose that H can be colored by three colors so that only colors a and b are used for horizontal paths.

The row r_1 is used by two paths which are colored interchangeably: either $a - b$ or $b - a$. Similarly, the row r_2 is used by three paths which are colored interchangeably: either $a - b - a$ or $b - a - b$; and the row r_3 is used by five paths which are colored interchangeably: either $a - b - a - b - a$ or $b - a - b - a - b$. For any way to color these rows and for any independent choice of colors $x_1, x_2, x_3 \in \{a, b\}$, there is a column among c_1, \dots, c_8 that meets r_1 in a point of a horizontal path colored by x_1 , meets r_2 in a point of a horizontal path colored by x_2 , and meets r_3 in a point of a horizontal path colored by x_3 .

In particular, there is a column c_i among c_1, \dots, c_8 that meets r_1 in a point of a horizontal path colored by a , meets r_2 in a point of a horizontal path colored by b , and meets r_3 in a point of a horizontal path colored by a . And there is a column c_j among c_1, \dots, c_8 that meets r_1 in a point of a horizontal path colored by b , meets r_2 in a point of a horizontal path colored by a , and meets r_3 in a

point of a horizontal path colored by b . A typical situation is shown in Figure 9, where c_i is c_1 and c_j is c_6 .

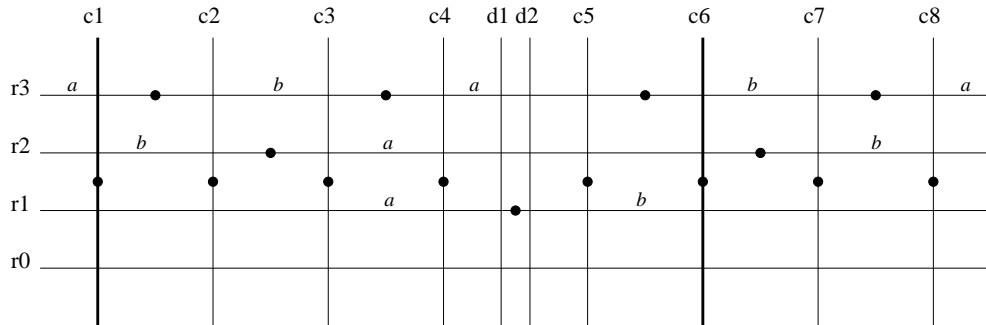


Figure 9: A typical bi-coloring of the horizontal lines in the B_0 -VPG representation of H .

Each of the columns c_i and c_j is used by two paths. The upper path of c_i intersects horizontal path colored by a and b , and therefore must be colored by c . Now the lower path of c_i intersects a horizontal path colored by a and it also intersects its upper path, therefore it is colored by b .

Similarly, the upper path of c_j is colored by c , and the lower path of c_j is colored by a .

Finally, at least one of d_1, d_2 meets a horizontal path colored by a and a horizontal path colored by b , and therefore it is colored by c .

Therefore, r_0 intersects vertical paths of all three colors a, b, c , and it is impossible to color it.

So we proved that if H is colored by three colors, all of them are needed for horizontal paths.

Step 2. Consider now the graph G represented in Figure 10 (the shaded regions are copies of the B_0 -VPG representation of H).

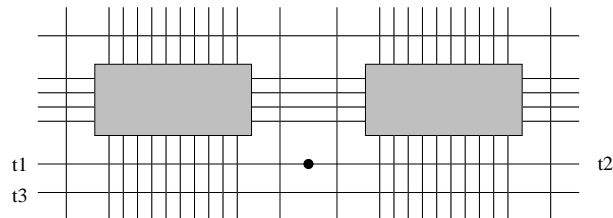


Figure 10: A representation of the graph G .

Consider a coloring of G by three colors a, b, c . The paths t_1 and t_2 must be colored by different colors. Therefore, without loss of generality, the color of t_1

is different from the color of t_3 : assume that t_3 is colored by a and t_1 is colored by b . However, we proved that for any coloring of H by three colors, each color is used by some horizontal path. Therefore, there is a horizontal path colored by c in the left (under our assumption) copy of H . Adding vertical paths in the neighborhoods of all vertical columns used by now, we see that one of them must then meet horizontal paths colored by a , b and c . Therefore, the graph G' (obtained from G by adding vertical paths, see Figure 11) cannot be colored by three colors. \square

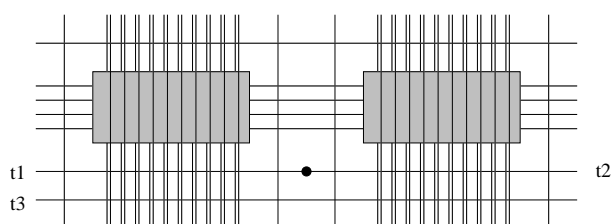


Figure 11: A representation of the graph G' .

9 Open Problems

In this paper, our focus has been primarily to study the B_k -VPG graphs, for $k = 0, 1$. We showed that the family of B_0 -VPG is strictly contained in the family of B_1 -VPG, and gave examples of VPG graphs that are not contained in B_1 -VPG. A general open question is whether the family of B_k -VPG is strictly contained in the family of B_{k+1} -VPG, for $k \geq 1$, which we expect is the case.

It is easy to see that the class B_0 -VPG is strictly contained in the class of graphs having boxicity two, with the graph W_4 as a separating example. We leave as an open question the connection between boxicity and the bending number of a graph.

Each path in a B_1 -VPG representation has one of four possible shapes: Right-Up, Right-Down, Left-Up and Left-Down. Which B_1 -VPG graphs are representable using only a proper subset of these shapes?

A pair of single-bend paths may intersect in exactly one grid point, in exactly two grid points or on a common subpath. Which B_1 -VPG graphs are representable using only a proper subset of these three possibilities?

In Theorem 7, we showed that every planar graph is in B_3 -VPG. We pose the question of finding a planar graph that is not in B_2 -VPG, thus making the result tight.

We saw in Theorem 9 that $\chi(G) \leq 2 \cdot \omega(G)$ for any B_0 -VPG graph G , and in Theorem 10 that this bound was tight for triangle-free graphs ($\omega = 2$). Is this bound also tight when $\omega = 3$ or for larger values of ω ?

The class of B_0 -VPG graphs, where every pair of paths can intersect in at most one grid point, is a subclass of B_0 -VPG graphs and a superclass of GIG.

One of the referees posed the question of determining the complexity of coloring such graphs, as our reduction from Theorem 8 does not hold for this case. Note, that by Theorem 9 the chromatic number of such class of graphs is at most eight.

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