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Switch-Regular Upward Planarity Testing of Directed Trees

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Abstract

Upward planar drawings of digraphs are crossing free drawings where all edges flow in the upward direction. The problem of deciding whether a digraph admits an upward planar drawing is called the *upward planarity testing problem*, and it has been widely studied in the literature. In this paper we investigate a new upward planarity testing problem, that is, deciding whether a digraph admits an upward planar drawing having some special topological properties: such a drawing is called *switch-regular*. Switch-regular upward planar drawings have practical algorithmic impacts in several graph drawing applications. We provide characterizations for the class of directed trees that admit a switch-regular upward planar drawing. Based on these characterizations we describe an optimal linear-time testing and embedding algorithm.

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1 Introduction

An *upward drawing* of a digraph is a drawing such that each vertex is mapped to a distinct point of the plane and all edges are drawn as simple Jordan curves monotonically increasing in the vertical direction. Upward drawings have a long tradition in Graph Drawing and they are commonly adopted for the visual representation of acyclic digraphs that model hierarchical structures, like PERT diagrams or class inheritance diagrams. A digraph is said to be *upward planar* if it admits an *upward planar drawing*, i.e., a crossing free upward drawing.

Although it is immediate to see that every acyclic digraph has an upward drawing, it is well known that not all acyclic planar digraphs are upward planar. Since edge crossings negatively affect the readability of a drawing, a very rich body of research has been devoted so far to the so called *upward planarity testing problem* (i.e., the problem of deciding whether or not a planar digraph admits an upward planar drawing), and many combinatorial and algorithmic results have been described (see, e.g., [8]). In particular, Bertolazzi *et al.* proved that given an embedded planar digraph G with n vertices, it is possible to decide in $O(n^2)$ time if G admits an embedding preserving upward planar drawing [2]. Conversely, Garg and Tamassia proved that the upward planarity testing problem in the variable embedding setting is NP-complete [12]. In the middle of these two results, polynomial-time upward planarity testing algorithms have been described for specific sub-families of planar digraphs [3, 11, 14, 15] and more general exponential-time algorithms can be found in [1, 6, 13].

Concerning the topological properties of upward planar drawings, Di Battista and Liotta discovered and characterized a meaningful sub-family of upward planar drawings whose embedding has some special properties of "regularity" [9]. Namely, an upward planar drawing has a *switch-regular* embedding if: (i) the boundary of each internal face contains at most one maximal subsequence of "small" angles (i.e., angles smaller than π) of length greater than one; (ii) the external boundary does not contain two consecutive "small" angles. Figure 1 shows examples of switch-regular and non switch-regular embeddings.

From a practical point of view, finding switch-regular upward planar embeddings is relevant for two main applications:

- *Design of Efficient Checkers.* A *checker* is an algorithm that efficiently checks the correctness of the output produced by another algorithm (see, e.g., [7]). Di Battista and Liotta showed that it is possible to design an optimal linear-time checker that verifies the correctness of a computed upward planar drawing, provided that its embedding is switch-regular. Namely, suppose we are given an algorithm that takes as input a planar digraph G and that computes, if any, an upward planar drawing Γ of G; if the embedding of Γ is switch-regular, the checker described by Di Battista and Liotta efficiently verifies the correctness of Γ in terms of upward planarity.
- *Effective Drawing Compaction.* Area and aspect ratio are considered two of the most important aesthetic requirements for the readability of a draw-

ing. There are works that experimentally show how, starting from a switch-regular embedding of a digraph or augmenting a non-switch regular embedding to a switch-regular one, it is possible to design heuristics that compute drawings with compact area and that dramatically improve aspect ratio with respect to previous drawing approaches [10]. We also remark that similar heuristics, based on a analogous concept of "regularity", were previously adopted and successfully experimented for the computation of orthogonal drawings [5].

These applications naturally motivate the following new upward planarity testing problem: *"Given a planar digraph G, is it possible to test in polynomial time whether G admits a switch-regular upward planar embedding?"*. In this paper we solve the problem for digraphs whose underlying undirected graph is a tree (we call such a digraph a *directed tree* for short). We remark that a directed tree always admits an upward planar embedding, but it may not admit a switch-regular one. Also, since an embedding of a tree has only one face (i.e., the external one) the switch-regularity reduces to the requirement that there are no two consecutive "small" angles along the face boundary. For example, Figure 7(a) shows a tree that does not admit a switch-regular embedding. Our results are as follows:

- We provide three equivalent characterizations of switch-regular directed trees, i.e., directed trees that admit a switch-regular upward planar embedding.
- By exploiting the above characterizations, we describe an optimal linear-time algorithm that tests whether a directed tree is switch-regular and that computes a switch-regular upward planar embedding of the tree in the affirmative case.

We remark that, besides their practical relevance, our techniques make use of new theoretical ingredients that are interesting in their own right. The outline of the paper is as follows. In Sections 3-5 we investigate the structure of switch-regular trees. Namely, in Section 3 we show that a switch-regular tree does not contain subdivisions of a special graph that we call a *3-hook* (see Lemma 1). In Section 4, we introduce *red-blue decompositions* of trees and we show that if a tree does not contain 3-hook subdivisions then it has a special kind of red-blue decomposition, which we call *regular* (see Lemma 5). In Section 5, we show that a regular red-blue decomposition of a tree T implies that T is switch-regular (see Lemma 10). In Section 6, we give different characterizations of switch-regular trees and describe a linear-time algorithm to test if a directed tree is switch-regular and, in positive case, to compute a switch-regular upward planar embedding of it. Conclusions and open problems can be found in Section 7.

2 Basic Definitions

We assume familiarity with the basic concepts of graph drawing and graph planarity [8]. Let G be an embedded planar digraph. A vertex v of G is *bimodal* if all incoming edges of v (and hence all outgoing edges of v) are consecutive in the circular clockwise order around v. G is called *bimodal* if all its vertices are bimodal. Acyclicity and bimodality are necessary conditions for the upward planar drawability of an embedded planar digraph, but they are not sufficient conditions [2].

Let f be a face of G and suppose that the boundary of f is traversed counterclockwise. If $e_1 = (u_1, v)$ and $e_2 = (v, u_2)$ are two edges encountered in this order along the boundary of f, the triplet $s = (e_1, v, e_2)$ is called an *angle of* f. Note that, e_1 and e_2 may coincide if G is not biconnected. Angle s is called a *switch of* f if e_1 and e_2 are both incoming edges or both outgoing edges of v: in the first case s is also called a *sink-switch of* f, while in the second case it is a *source-switch of* f. Observe that the number of source-switches of f equals the number of sink-switches of f. We denote by $2n_f$ the total number of switches of f. The *capacity of* f is denoted by c_f and it is defined by $c_f = n_f - 1$ if f is an internal face and by $c_f = n_f + 1$ if f is the external face. If G is bimodal an assignment of the sources and sinks of G to the faces of G is called an *upward consistent assignment* of G if, for each face f exactly c_f sources and sinks on the boundary of f are assigned to f. The following theorem gives a characterization of the class of embedded planar digraphs that admit an upward planar drawing.

Theorem 1 [2] *An acyclic embedded planar bimodal digraph is upward planar if and only if it admits an upward consistent assignment.*

The *upward planar embedding* of G corresponding to an upward consistent assignment of G is a planar embedding of G with labels at the switches of every face. Namely, a switch $s = (e_1, v, e_2)$ of f is labeled L when v is a source or a sink assigned to f and s is labeled S otherwise. If f is a face of an upward planar embedding, the circular list of labels of f is denoted by σ_f. Also, S_{σ_f} and L_{σ_f} denote the number of S and L labels of f, respectively.

Property 1 [2] *If f is a face of an upward planar embedding then $S_{\sigma_f} = L_{\sigma_f} + 2$ if f is internal, and $S_{\sigma_f} = L_{\sigma_f} - 2$ if f is external.*

An upward planar drawing of a digraph G can be constructed for a given upward planar embedding of G; this drawing is such that each switch angle labeled L forms a geometric angle larger than π, while each switch angle labeled S forms a geometric angle smaller than π.

An internal face f of an upward planar embedding is called *switch-regular* if σ_f does not contain two distinct maximal subsequences σ_1 and σ_2 of S labels such that $S_{\sigma_1} > 1$ and $S_{\sigma_2} > 1$. The external face f is *switch-regular* if σ_f does not contain two consecutive S labels.

An upward planar embedding is *switch-regular* if all its faces are switch-regular. We say that a digraph G is *switch-regular* if it admits a switch-regular

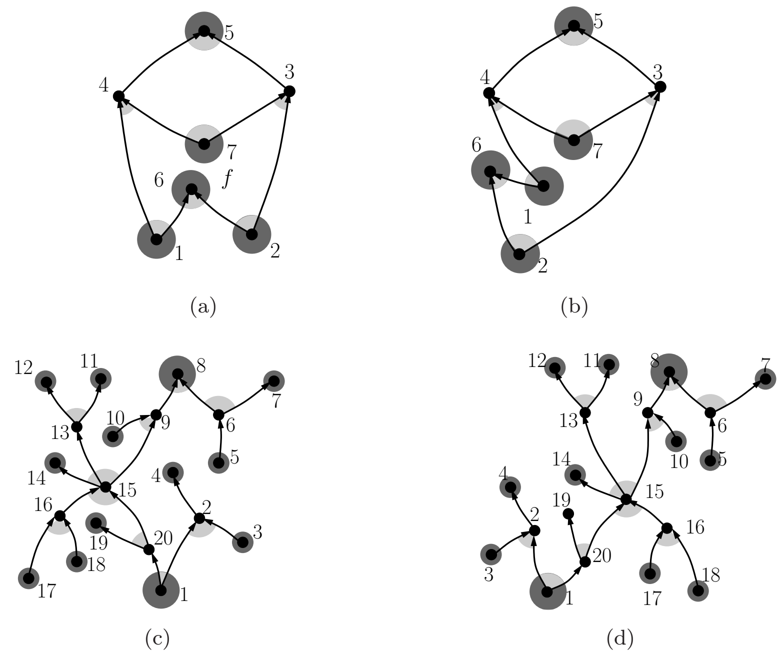


Figure 1: (a) and (b) show two different embeddings of the same digraph G, while (c) and (d) show two different embeddings of the same directed tree T. Dark angles are switches labeled L, light angles are switches labeled S. In (a) the embedding of G is not switch-regular because the circular list of labels along the boundary of face f has two distinct subsequences, each containing two consecutive S labels (the labels at vertices 3 and 2 in a subsequence, and the labels at vertices 1 and 4 in the other subsequence). In (b) the embedding of G is switch-regular because all its faces are switch-regular. Similarly, in (c) the embedding of T is not switch-regular because the external face (which is the unique face of T) is not switch-regular; for example there are two consecutive S labels at vertices 15 and 9 or at vertices 15 and 20. In (d) the external face does not contain consecutive S labels, and thus the embedding is switch-regular.

upward planar embedding. Examples of switch-regular and non switch-regular upward planar embeddings are depicted in Figure 1. Roughly speaking, a switch-regular embedding does not have a pair of vertices that are "facing" one each other in some face (like the vertices 6 and 7 in face f of Figure 1(a) or the vertices 4 and 5 in Figure 1(c)).

If $e = (u, v)$ is a directed edge of a digraph G, a *subdivision* of e is a path of directed edges (u, w_1), $(w_1, w_2) \dots (w_k, v)$ that replaces e $(k > 0)$. A digraph obtained from G by subdividing some edges (possible none) of G is called a *subdivision* of G.

In the next sections we investigate the structure of switch-regular trees in order to characterize them. Intuitively speaking, throughout the paper we will prove that a tree is switch-regular if it admits an embedding like that schematically depicted in Figure 2. Namely, if a tree is switch-regular it should be possible to select a vertex v such that: (i) there are vertices that can either be reached from v or reach v by means of a directed path (they induce the blue portions of the tree shown in the figure, and form a kind of "hourglass" shape); (ii) the remaining vertices form components attached to paths of blue vertices (the red components in the figure); (iii) all the red components can be externally embedded as shown in Figure 2(a) or in Figure 2(b) (or other symmetric fashions).

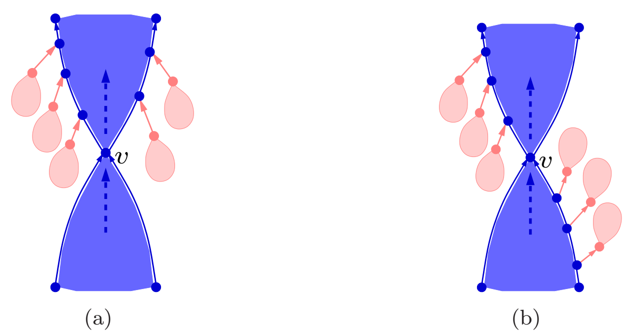


Figure 2: Schematic illustration of switch-regular embedded trees. The blue portions represent vertices that can either be reached from v or reach v by means of a directed path. The remaining vertices form components (in red) that can be externally embedded in a fashion like those depicted in the two subfigures.

3 Switch-regularity and 3-hooks

In this section we introduce a special graph that we call 3-*hook* and we show that a switch-regular directed tree cannot contain subdivisions of this graph.

A *hook* is a digraph H whose underlying undirected graph is a path consisting of three vertices such that the middle vertex is either a source or a sink of H. A 3-*hook* is a directed tree consisting of three hooks sharing an endvertex. The vertex shared by the three hooks is called the *center of the* 3-*hook*; the middle vertex of each hook is called a *middle vertex of the* 3-*hook*; the unshared endvertex of a hook is called a *leaf of the* 3-*hook*. Figure 3 shows 4 different 3-hook graphs. A subdivision of a hook is called a *hook subdivision* and a subdivision of a 3-hook is called a 3-*hook subdivision*. The *center*, the *middle vertices*, and the *leaves* of a 3-hook subdivision are the vertices corresponding to the center the middle vertices, and the leaves of the 3-hook. Note that, by definition, a subdivision does not create new sources and sinks, and hence there is no ambiguity about the location of the middle vertices. In any upward embedding of a 3-hook, every middle vertex of the 3-hook defines 2 switches, one of which is labeled L. Every leaf of the 3-hook defines one switch that is always labeled L.

It is easy to see that any 3-hook is not switch-regular, because in any given embedding of a 3-hook there are always two consecutive S-labels. In the next lemma we prove that if a directed tree is switch-regular, then it does not contain a 3-hook subdivision. For the proof it is sufficient to show that every 3-hook subdivision induces two consecutive S-labels.

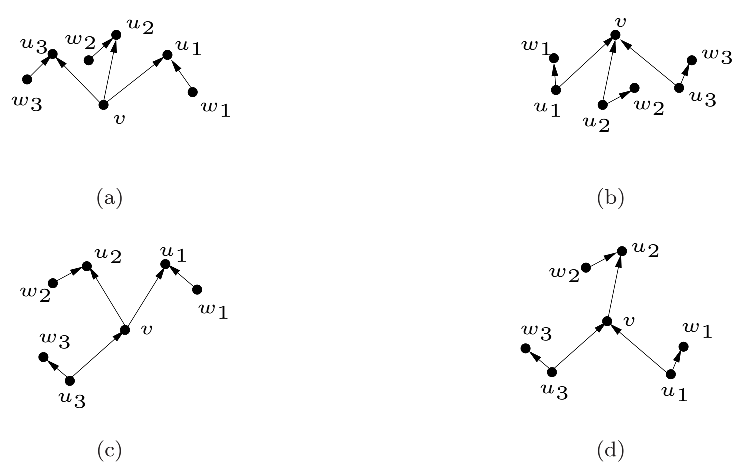


Figure 3: Four different 3-hooks. In each of them, v is the center, vertices u_i are the middle vertices, and vertices w_i are the leaves ($i \in \{1, 2, 3\}$).

Lemma 1 *A switch-regular directed tree does not contain* 3*-hook subdivisions.*

Proof: We assume by contradiction that T contains a subtree T' that is a 3-hook subdivision. We will show that T is not switch-regular in this case. Let

ψ be an upward planar embedding of T, and let v be the center of T'. Assume that H is any of the three subgraphs of T' that are hook subdivisions, and let u denote the middle vertex of H. We say that H is an *incoming hook* (*outgoing hook*) if there is a directed path from u to v (v to u). Let ψ' be the upward planar embedding of T' induced by ψ; one of the two switches at u is labeled S in ψ' while the other is labeled L. We say that H is a *left hook* if walking counterclockwise around T', starting from v, the switch labeled L incident to u is encountered before the switch labeled S, while H is a *right hook* otherwise.

One of the two following cases always holds for ψ':

Case 1. There is an incoming and an outgoing hook such that one is a left hook and the other one is a right hook. Let us assume that H_1 is an outgoing left hook and H_2 is an incoming right hook (see Figure 4(a)). The other cases are symmetric. Let u_1 and u_2 be the middle vertices of H_1 and H_2, respectively. Also let e_1 be the edge incident to u_1 in the path from v to u_1, and let e'_1 be the other edge of H_1 incident to u_1. Since H_1 is a left hook, $\hat{s} = (e'_1, u_1, e_1)$ is labeled S. Analogously, let e_2 be the edge incident to u_2 in the path from u_2 to v, and let e'_2 be the other edge of H_2 incident to u_2. Since H_2 is a right hook, $\bar{s} = (e_2, u_2, e'_2)$ is labeled S.

Case 2. There are two incoming or two outgoing hooks, such that both are either left hooks or right hooks. We consider the case when there are two outgoing hooks H_1 and H_2 that are both left hooks (see Figure 4(b)). The other cases are symmetric. Let e_i be the edge of H_i ($i = 1, 2$) incident to v, and let e'_i be the edge that follows e_i in the counterclockwise order around v. One between H_1 and H_2 is such that (e_i, v, e'_i) is labeled S. Assume without loss of generality that such a hook is H_1 and denote as $\bar{s}$ the switch (e_1, v, e'_1). Let u be the middle vertex of H_1 and let e be the edge incident to u in the path from v to u. Let e' be the edge of H_1 incident to u and distinct from e. Since H_1 is a left hook, $\hat{s} = (e', u, e)$ is labeled S.

Both **Case 1** and **Case 2** have four subcases. Let $\Pi = H_1 \cup H_2$ in **Case 1** and let $\Pi = H_1$ in **Case 2**.

Sub-Case 1. No switch is encountered when walking counterclockwise around T from $\hat{s}$ to $\bar{s}$. In this case $\hat{s}$ and $\bar{s}$ form a sequence of two consecutive S labels and therefore T is not switch-regular.

Sub-Case 2. Walking counterclockwise from $\hat{s}$ to $\bar{s}$ the first switch $s_3 = (e_3, w, e'_3)$ encountered after $\hat{s}$ is such that e_3 is an edge of Π, w is a vertex of Π, and e'_3 leaves w. The switches $\hat{s}$ and s_3 form a sequence of two consecutive S labels because s_3 is also labeled S.

Sub-Case 3. Walking counterclockwise from $\hat{s}$ to $\bar{s}$ the last switch $s_3 = (e'_3, w, e_3)$ encountered before $\bar{s}$ is such that e_3 is an edge of Π, w is a vertex of Π, and edge e'_3 enters w. The switches s_3 and $\bar{s}$ form a sequence of two consecutive S labels because s_3 is also labeled S.

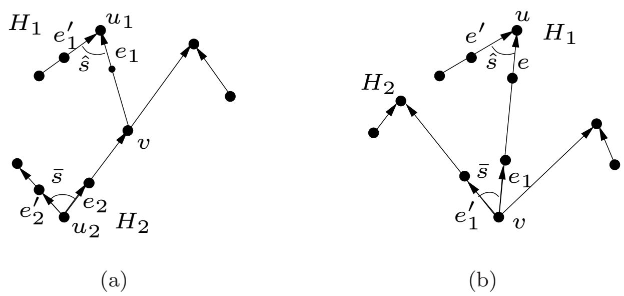


Figure 4: Illustration for Lemma 1: (a) **Case 1**. Starting from v, H_1 is an outgoing left hook because the switch labeled L at vertex u_1 is encountered before the switch $\hat{s}$ labeled S; conversely, H_2 is an incoming right hook because the switch $\bar{s}$ labeled S is encountered before the switch labeled L at vertex u_2; (b) **Case 2**: Both H_1 and H_2 are left outgoing hooks.

Sub-Case 4. Walking counterclockwise from $\hat{s}$ to $\bar{s}$ there are two switches $s_3 = (e_3', w_3, e_3)$ and $s_4 = (e_4, w_4, e_4')$ such that e_i is an edge of Π $(i = 3, 4)$, w_i is a vertex of Π $(i = 3, 4)$, e_3' is an edge entering w_3 and e_4' is an edge leaving w_4. Both s_3 and s_4 are labeled S. If they are consecutive they form a sequence of two consecutive S labels, otherwise walking counterclockwise from s_3 to s_4 there are two consecutive switches with the same properties as s_3 and s_4. These switches form a sequence of two consecutive S labels.

□

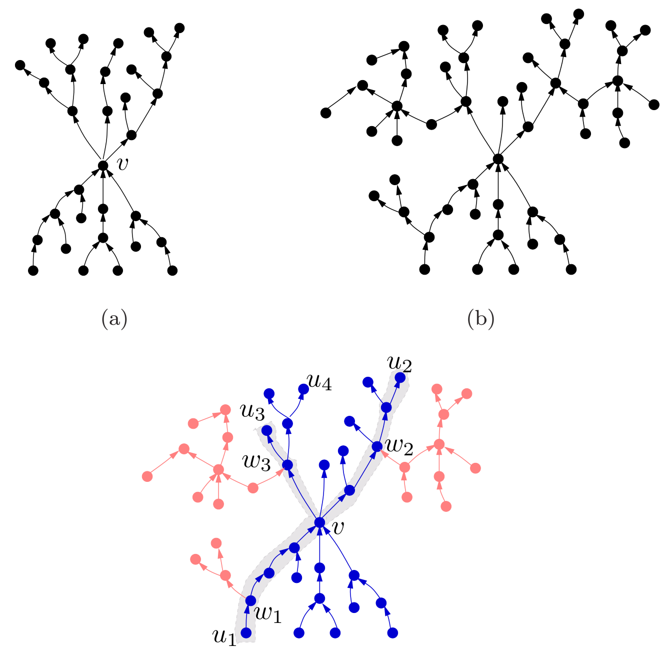
Observe that if T has vertex degree at most 2 then T is a path, which always admits a switch-regular embedding. Thus, from now on we concentrate on trees with at least one vertex with degree larger than 2.

4 3-hooks and Red-blue Decompositions

In this section we introduce the concept of *red-blue decomposition* of a directed tree and we show the relationship between it and 3-hook subdivisions. In particular, we show that if a tree does not contain 3-hook subdivisions then it has a special kind of red-blue decomposition, which we call *regular* (see Lemma 5). To this aim we describe three properties that a red-blue decomposition must satisfy in order to be regular (see Lemmas 2, 3, 4).

An *hourglass tree* T is a directed tree with a vertex v such that for each vertex u of T $(u \neq v)$ either there is a directed path from v to u or there is a directed path from u to v. The vertex v is called the *center of the hourglass*. See Figure 5(a) for an example of an hourglass tree.

Property 2 *Every upward planar embedding of an hourglass tree is switch-regular.*



(a) (b)

(c)

Figure 5: (a) An hourglass tree with center v: Each vertex "above" v is reachable from v with a directed path and each vertex "below" v can reach v with a directed path; (b) A tree T that is not an hourglass tree. Indeed, for any selected vertex v of T, there exists at least one vertex that cannot reach v or that it is not reachable from v by means of a directed path. (c) A red-blue decomposition of a tree T with respect to v; the directed blue path from u_1 to v is an incoming attaching path of $RB(T, v)$ and w_1 is its last attaching vertex. The directed blue paths from v to u_2 and from v to u_3 are both outgoing attaching paths of $RB(T, v)$; w_2 and w_3 are their corresponding last attaching vertices.

Let T be a directed tree and let v be a vertex of T with $deg(v) \geq 3$. A *red-blue decomposition* of T with respect to v is a coloring of the vertices and edges of T such that: (i) a vertex u of T is colored *blue* if there exists a directed path either from u to v or from v to u, and u is colored *red* otherwise; (ii) an edge e of T is colored *blue* if both its endvertices are blue, and e is colored *red* otherwise. We denote by $RB(T, v)$ such a decomposition. If e is a red edge of $RB(T, v)$, then either both end-vertices of e are red or one is red and the other is blue. By definition the subgraph consisting of all blue vertices is an hourglass tree.

Let u and w be a red and a blue vertex of $RB(T, v)$, respectively. We say that u is *attached* to w if there exists a (non-directed) path from u to w whose vertices are all red vertices except w. We also say that w is the *attaching vertex* of u.

An *outgoing (incoming) attaching path of* $RB(T, v)$ is a directed blue path Π from v to a leaf of T (from a leaf of T to v) such that at least one vertex of Π is an attaching vertex (see Figure 5(c)). An *attaching path of* $RB(T, v)$ is either an outgoing or an incoming attaching path with respect to v. Given an attaching path Π we call *last attaching vertex* of Π the attaching vertex that has maximum distance from v. Let Π_1 and Π_2 be two attaching paths. We say that Π_1 and Π_2 are *distinct* if their last attaching vertices are distinct and none of them is shared by Π_1 and Π_2. For example, the two attaching paths from v to u_3 and from v to u_4 in Figure 5(c) are not distinct because they share the last attaching vertex w_3. Clearly, an outgoing and an incoming attaching paths are always distinct. We say that Π_1 and Π_2 are *equally oriented* if they are both incoming or both outgoing.

Concerning the number of distinct attaching paths of $RB(T, v)$ we give the following result.

Lemma 2 *Let T be a directed tree having a vertex v with $deg(v) \geq 3$ such that $RB(T, v)$ has more than two distinct attaching paths. Then T contains a 3-hook subdivision.*

Proof: Let Π_1, Π_2, and Π_3 be three distinct attaching paths of $RB(T, v)$. Let u_i be an attaching vertex of Π_i not shared with another attaching path Π_j and let e_i be a red edge incident to u_i $(1 \leq i \neq j \leq 3)$. Let w_i be the last vertex (i.e., the farthest from v) of Π_i shared with another attaching path Π_j and let Π'_i be the portion of Π_i from w_i to u_i $(1 \leq i \neq j \leq 3)$. We have the following cases. For an illustration see Figure 6.

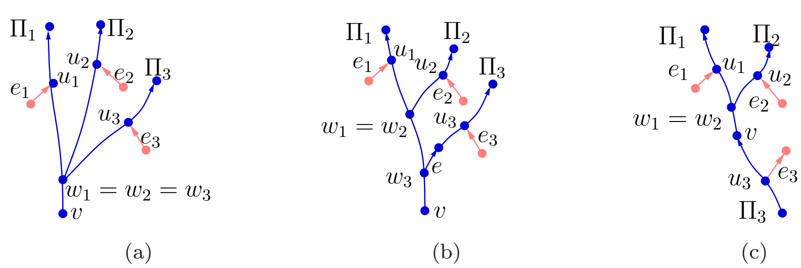


Figure 6: Illustration for Lemma 2. (a) **Case 1**: $w_1 = w_2 = w_3$; there exists a 3-hook subdivision with center w_1. (b)-(c) **Case 2**: In (b) Π_1, Π_2 and P_3 are equally oriented; (c) Π_3 is not equally oriented with Π_1 and Π_2. In both cases there is a 3-hook subdivision with center w_1.

Case 1: $w_1 = w_2 = w_3$. Notice that $w_1 = w_2 = w_3$ may coincide with v. In this case $\Pi'_i \cup e_i$ is a hook subdivision with w_i as an end-vertex $(1 \leq i \leq 3)$. Hence, there is a 3-hook subdivision with center w_1.

Case 2: $w_1 = w_2$ Notice that $w_1 = w_2$ does not coincide with v and therefore Π_1 and Π_2 are equally oriented. In this case $\Pi'_i \cup e_i$ is a hook subdivision with w_i as an end-vertex ($1 \leq i \leq 2$). Let Π^* be the path from w_1 to w_3. If Π_3 is equally oriented with Π_1 and Π_2, let e be the edge of Π'_3 incident to w_3. Then $\Pi^* \cup e$ is a hook subdivision with w_1 as an endvertex and therefore we have a 3-hook subdivision. If Π_3 is not equally oriented with Π_1 and Π_2, then $\Pi^* \cup e_3$ is a hook subdivision with w_1 as an endvertex and therefore we have a 3-hook subdivision.

□

From Lemma 2 we know that $RB(T, v)$ must contain at most two distinct attaching paths, otherwise T contains a 3-hook subdivision. This condition is necessary but not sufficient to avoid the presence of 3-hook subdivisions. Now we describe a second condition that is related with the concept of *regular red component*.

Let $C' = (V_{C'}, E_{C'})$ be any connected component obtained by removing the blue vertices. Note that all vertices of C' have the same attaching vertex w. Let e be the (unique) edge of T that connects w to C'. The subtree $C = (V_{C'} \cup \{w\}, E_{C'} \cup \{e\})$ is called a *red component of* $RB(T, v)$ and vertex w is called the *attaching vertex* of the red component C (see Figure 7(a)). We say that C is *regular* if it contains a (non-directed) path Π having w as an endvertex and such that C minus the edges of Π is a forest of hourglass trees whose centers belong to Π. If C is regular, we call Π a *backbone* of C. We say that C is *strongly regular* if the path consisting of the only vertex w is a backbone of C (in this case no edge is removed from C). In other words, C is strongly regular, if either all vertices of C are reachable with a directed path from w, or w is reachable with a directed path from all vertices of C. If C is regular but not strongly regular, it is said to be *weakly regular*. Figure 7(a) shows examples of regular and non-regular red components in a red-blue decomposition of a directed tree.

The following lemma shows that a non-regular red component in $RB(T, v)$ induces a 3-hook subdivision in T.

Lemma 3 *Let T be a directed tree having a vertex v with $deg(v) \geq 3$ such that $RB(T, v)$ has a non-regular red component. Then T contains a 3-hook subdivision.*

Proof: Let C be a red component of $RB(T, v)$ that is not regular. We observe that the following property holds:

Property 3 *Let T be a tree and let z be a vertex of T. If T is not an hourglass with center z, then T has a subtree that is a hook subdivision with z as an endvertex.*

Let w be the attaching vertex of C. Let Π be any (non-directed) path of C having w as an endvertex. Since C is not regular, removing the edges of Π we obtain at least one tree that is not an hourglass with its center on Π; we

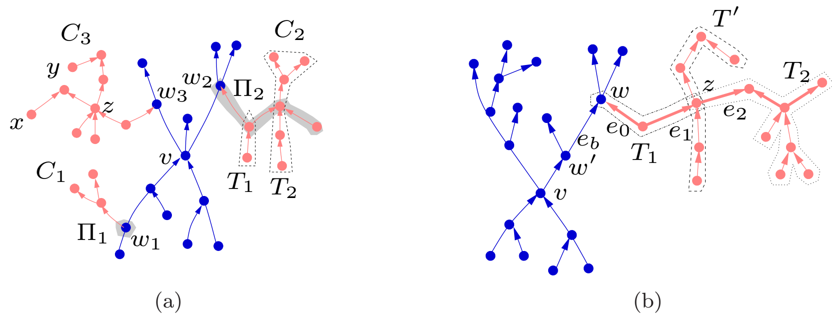


Figure 7: (a) A red-blue decomposition of a tree T with respect to v; C_1, C_2, and C_3 are red components of $RB(T,v)$ and w_1, w_2, and w_3 are their corresponding attaching vertices. C_1 is strongly regular: Its backbone Π_1 consists only of the attaching vertex w_1. C_2 is weakly regular with Π_2 as backbone: The removal of the edges of Π_2 will keep alive the two hourglass trees T_1 and T_2 with centers on Π_2. C_3 is not regular, because there is no directed path having w_3 as an endvertex that can be the backbone of C_3. For example, the path from w_3 and x is not a backbone, because y is not reachable with a directed path from z; similarly the path from w_3 and x is not a backbone because x is not reachable from z. (b) Illustration for Lemma 3: A red component C of $RB(T,v)$ that has w as attaching vertex and that is not regular. Bold red lines represent the path Π of C having w as an endvertex.

call such a tree a *candidate tree*. Let T' be the first candidate tree encountered while walking along Π from w. Let z be the vertex shared by T' and Π; we call this vertex the *reference vertex*. Since T' is not an hourglass with center z, by Property 3 T' contains a hook subdivision H_1 with z as an endvertex. Let e_1 be the edge of Π incident to z encountered first when walking along Π starting from w and let e_2 be the other edge of Π incident to z (refer to Figure 7(b)). Also, let e_0 denote the edge of Π incident to w (note that e_0 and e_1 may coincide). Let T_i be the connected subtree of C obtained by removing all edges incident to z except e_i ($i = 1, 2$) and containing z. Since w is a blue vertex of $RB(T,v)$, then there exists a blue edge e_b of $RB(T,v)$ incident on w having the same orientation as e_0. Let w' be the other vertex incident to e_b. Tree $T_1 \cup e_b$ is not an hourglass with center z because otherwise there would be a directed path from w' to z, which is impossible because z is a red vertex of $RB(T,v)$. Therefore $T_1 \cup e_b$ contains a hook subdivision H_2 with z as an endvertex. If T_2 is not an hourglass with center z then it contains a hook subdivision H_3 with z as an endvertex and therefore T has a subgraph that is a 3-hook subdivision. If T_2 is an hourglass with center z, consider the path $\overline{\Pi} = \Pi' \cup H_1$ where Π' is the portion of Π with endvertices w and z. Removing the edges of $\overline{\Pi}$ we obtain a new set of candidate trees. Let $\overline{T'}$ be the first candidate tree of this new set. Let $\overline{z}$ be the new reference vertex, i.e., the vertex shared by $\overline{\Pi}$ and $\overline{T'}$. If $\overline{z} = z$, then $\overline{T'}$ has a hook-subdivision H_3 with z as an endvertex that is distinct from H_1 and H_2, i.e., T has a 3-hook subdivision. If $\overline{z} \neq z$, then $\overline{z}$ is farther from w than z. We can apply the same argument used for Π and T' to $\overline{\Pi}$ and $\overline{T'}$. Clearly, it may happen that we are in this case again and therefore we have to repeatedly consider new paths and new sets of candidate trees and a vertex farther from w becomes the reference vertex. It follows that either at some point we are no longer in this case and there is a 3-hook subdivision, or a leaf becomes the reference vertex. However, a leaf cannot be a reference vertex because otherwise it should have more than one incident edge. $\square$

Unfortunately, even when $RB(T, v)$ has at most two distinct attaching paths and all its red components are regular, there are cases in which T may still contain a 3-hook subdivision. To introduce these cases we need some further definitions.

Let u be a vertex of an attaching path Π of $RB(T, v)$; we say that u is:

- a *k-regular* vertex ($k > 0$), if it is the attaching vertex of at least k regular red components; a k-regular vertex is also called *regular*.
- a *k-weak-regular* vertex ($k > 0$), if it is the attaching vertex of at least k weakly regular red components; a k-weak-regular vertex is also called *weak-regular*.
- a *branch* vertex, if it has two incident blue edges such that they are both entering/leaving u and one of them belongs to Π.
- a *branch attaching* vertex, if it is a branch shared by two distinct attaching paths.

Note that a k-weak-regular vertex is also k-regular and that a branch attaching vertex is also a branch vertex. Figure 8(a) shows examples of k-regular and k-weak-regular vertices. Figure 8(b) shows examples of branch or branch attaching vertices.

(a) (b)

Figure 8: Π_1 and Π_2 are two distinct attaching paths. (a) u_1, u_2, u_3, and u_4 are regular vertices; also, u_2 and u_4 are weak-regular vertices. (b) u_1 is a branch attaching vertex and u_2 is a branch vertex.

We say that v is *internal attaching* if it is shared by two attaching paths that are one incoming and one outgoing. In Figure 8(a) v is internal attaching.

Let $RB(T, v)$ be a red-blue decomposition of T with respect to v, such that all red components of $RB(T, v)$ are regular. A *forbidden configuration* of $RB(T, v)$ is one of the following configurations:

FC1. There exists an attaching path Π of $RB(T, v)$ such that walking along Π starting from v we encounter three not necessarily consecutive vertices u_1, u_2, u_3 (where u_1 may coincide with v) such that: u_1 is either weak-regular or branch or internal attaching, u_2 is weak-regular or branch attaching, and u_3 is regular or branch attaching. Figure 9 shows some examples of this forbidden configuration.

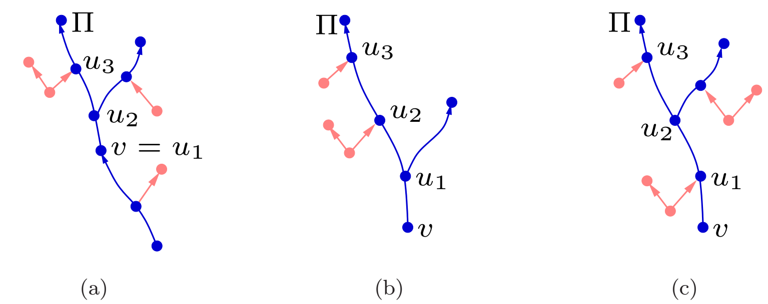


Figure 9: Examples of forbidden configurations of type **FC1**: (a) u_1 coincides with v and is internal attaching, u_2 is branch attaching and u_3 is regular; (b) u_1 is branch, u_2 is weak-regular and u_3 is regular; (c) u_1 is weak-regular, u_2 is branch attaching and u_3 is regular.

FC2. There exists an attaching path Π of $RB(T, v)$ such that walking along Π starting from v we encounter two not necessarily consecutive vertices u_1, u_2 (where u_1 may coincide with v) such that u_1 is either weak-regular or branch or internal attaching, and u_2 is either 2-weak-regular or weak-regular and branch attaching at the same time. Figure 10 shows some examples of this forbidden configuration.

FC3. There exists an attaching path Π of $RB(T, v)$ such that walking along Π starting from v we encounter two not necessarily consecutive vertices u_1, u_2 (where u_1 may coincide with v) such that u_1 is either 2-weak-regular or weak-regular and branch attaching at the same time, and u_2 is either regular or branch attaching. Figure 11 shows some examples of this forbidden configuration.

FC4. There exists one vertex that is either 3-weak-regular or 2-weak-regular and branch attaching at the same time. Figure 12 shows some examples of this forbidden configurations.

Lemma 4 *Let T be a directed tree having a vertex v with $deg(v) \geq 3$ such that $RB(T, v)$ has a forbidden configuration. Then T contains a 3-hook subdivision.*

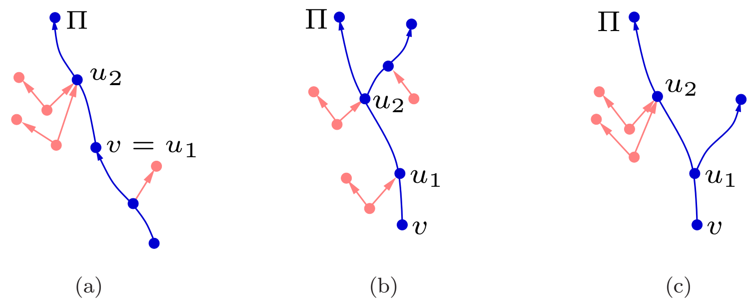


Figure 10: Examples of forbidden configurations of type **FC2**: (a) u_1 coincides with v and is internal attaching, u_2 is 2-weak-regular; (b) u_1 is weak-regular and u_2 is weak-regular and branch attaching at the same time; (c) u_1 is branch and u_2 is 2-weak-regular.

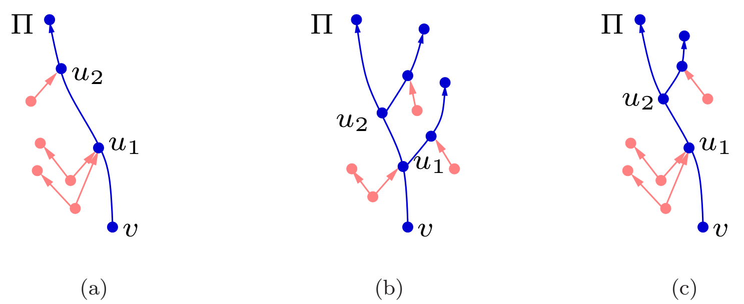


Figure 11: Examples of forbidden configurations of type **FC3**: (a) u_1 is 2-weak-regular and u_2 is regular; (b) u_1 is weak-regular and branch attaching at the same time and u_2 is branch attaching; (c) u_1 is 2-weak-regular and u_2 is branch attaching.

Figure 12: Examples of forbidden configurations of type **FC4**: (a) u is 3-weak-regular; (b) u is 2-weak-regular and branch attaching at the same time.

Proof: We start the proof by proving the following three facts. Let Π_1 be an attaching path. Assume that Π_1 is an outgoing attaching path, the other case is analogous.

Fact 1. *If u is a vertex of Π_1 that is either weak-regular or branch attaching, then there exists a hook subdivision H having u as an endvertex and such that no edge is shared between H and Π_1.* Refer to Figure 13(a) and 13(b). If u is weak-regular, let C be the weakly regular red component having u as its attaching vertex. C is not an hourglass with center u because otherwise it would be strongly regular. By Property 3, C contains a hook subdivision with u as an endvertex. If u is branch attaching, let Π_2 be the attaching path that shares u with Π_1. Let e be the edge incident to u that belongs to Π_2 but not to Π_1, and let T' be the connected subtree of T containing u and obtained by removing all edges incident to u except e. T' is not an hourglass with center u. Namely, consider the portion Π' of Π_2 that is contained in T'. Since Π_1 and Π_2 are distinct, there exists an attaching vertex u' that belongs to Π' and therefore to T'. Let u'' be a vertex of a red component attached to u'. There is not a directed path from u to u'' or from u'' to u because otherwise u'' would be blue. Hence, T contains a hook subdivision with u as an endvertex.

Fact 2. *Let u and u_1 be two vertices of Π_1 such that u_1 is encountered before u when walking along Π_1 starting from v. If u_1 is either weak-regular, or branch, or internal attaching, then there exists a hook subdivision H having u as an endvertex.* Refer to Figure 13(c) and 13(d). Let T' be the connected subtree of T containing u and obtained by removing all edges incident to u except the one entering u and belonging to Π_1. If u_1 is either weak-regular, or branch, or internal attaching, then T' is not an hourglass with center u. If u_1 is weak-regular then there exists a vertex u' of the weakly regular red component C attached to u_1 such that there is no directed path from u' to u because otherwise C would be a strongly regular red component. If u_1 is branch, then there is an outgoing edge $e = (u_1, u')$ incident to u_1 that does not belong to Π_1. Also in this case there is no directed path from u' to u. If u_1 is internal attaching, i.e., $u_1 = v$, there is an incoming attaching path Π_2. Let u' be a vertex of a red component attached to Π_2; there is no directed path from u' to u because otherwise u' would be blue. Hence in all the three cases T' contains a hook subdivision with u as an endvertex.

Fact 3. *Let u and u_2 be two vertices of Π_1 such that u_2 is encountered after u when walking along Π_1 starting from v. If u_2 is either regular or branch attaching, then there exists a hook subdivision H having u as an endvertex.* Refer to Figure 13(e) and 13(f). Let T' be the connected subtree of T containing u and obtained by removing all edges incident to u except the one leaving u and belonging to Π_1. We prove that if u_2 is either regular or branch attaching, then T' is not an hourglass with center u. If u_2 is regular then there exists a vertex u' of the regular red component C attached to

u_2 such that there is no directed path from u to u' because otherwise u' would be blue. If u_2 is branch attaching, then let Π_2 be the attaching path that shares u_2 with Π_1. Since Π_2 and Π_1 are distinct, there exists an attaching vertex u'' that belongs to Π_2 but not to Π_1. Let u''' be a vertex of a regular red component C attached to u''. There is no directed path from u to u''' because otherwise u''' would be blue. Hence, in both cases T' contains a hook subdivision with u as an endvertex.

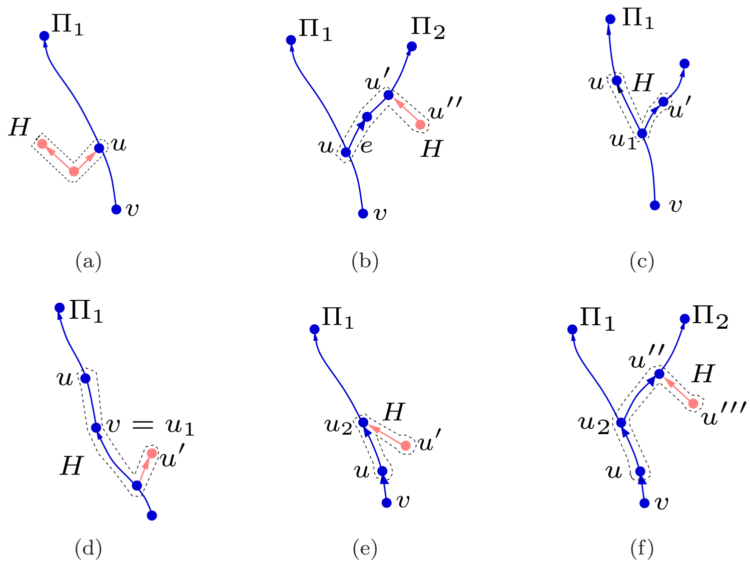


Figure 13: Configurations described in **Fact 1**, **Fact 2** and **Fact 3**. All these configurations induce a hook subdivision H having u as an endvertex. (a) and (b) refer to **Fact 1**: In (a) u is weak-regular, while in (b) u is branch attaching. (c) and (d) refer to **Fact 2**: In (c) u_1 is branch, while in (d) u_1 is internal attaching and coincides with v. (e) and (f) refer to **Fact 3**: In (e) u_2 is regular, while in (f) it is branch attaching.

If we have a forbidden configuration **FC1**, walking along Π_1 we encounter three vertices v_1, v_2 and v_3, in this order, such that v_1 corresponds to vertex u_1 in **Fact 2**, v_2 corresponds to u in **Fact 1**, **Fact 2**, and **Fact 3**, and v_3 corresponds to u_2 in **Fact 3**. Therefore there exist three hook subdivisions with v_2 as an endvertex, i.e., T contains a 3-hook subdivision.

If we have a forbidden configuration **FC2**, then we have two vertices v_1 and v_2, in this order, such that v_2 is either 2-weak-regular or weak-regular and branch attaching at the same time. By **Fact 1** there exists two hook subdivisions with v_2 as an endvertex. Also, v_1 corresponds to u_1 in **Fact 2** and v_2 corresponds to u **in Fact 2**. This implies that there is another hook subdivision with v_2 as an endvertex, i.e., T contains a 3-hook subdivision also in this case.

If we have a forbidden configuration **FC3**, then we have two vertices v_1 and

v_2, in this order, such that v_1 is either 2-weak-regular or weak-regular and branch attaching at the same time. By **Fact 1** there exists two hook subdivisions with v_1 as an endvertex. Also, v_2 corresponds to u_2 in **Fact 3** and v_1 corresponds to u **in Fact 3**. This implies that there is another hook subdivision with v_1 as an endvertex. Hence, T contains a 3-hook subdivision also in this case.

If we have a forbidden configuration **FC4**, then we have one vertex v_1 that is either 3-weak-regular or 2-weak-regular and branch attaching at the same time. By **Fact 1** there exists three hook subdivisions with v_1 as an endvertex, i.e., T contains a 3-hook subdivision also in this case. □

The next definition is motivated by Lemmas 2, 3, 4. Let T be a directed tree and let v be a vertex of T with $deg(v) \geq 3$. $RB(T, v)$ is said to be *regular* if the following conditions hold:

RB1. $RB(T, v)$ has at most two distinct attaching paths;

RB2. Every red component of $RB(T, v)$ is regular;

RB3. $RB(T, v)$ has no forbidden configuration.

The following lemma summarizes the main result of this section.

Lemma 5 *Let T be a directed tree with at least one vertex whose degree is larger than two. If T does not contain 3-hook subdivisions, then for every vertex v with $deg(v) \geq 3$ $RB(T, v)$ is regular.*

Proof: Assume by contradiction that there exists a vertex v with $deg(v) \geq 3$ such that $RB(T, v)$ is not regular. This implies that at least one of conditions **RB1-RB3** is violated. By Lemmas 2, 3, 4, it follows that T contains a 3-hook subdivisions, a contradiction. □

5 Red-blue Decompositions and Switch-regularity

So far we have proved that if T is switch-regular, then $RB(T, v)$ is regular, for any vertex v with $deg(v) \geq 3$. We now prove that the converse is also true; namely if $RB(T, v)$ is regular (for any chosen vertex v with $deg(v) \geq 3$), then T is switch-regular (Lemma 10). In order to prove Lemma 10, we describe a linear-time algorithm that computes a switch-regular embedding of T from $RB(T, v)$. Intuitively, since $RB(T, v)$ is regular, then it has at most two distinct attaching paths; the algorithm embeds one of the two attaching paths as the leftmost one and the other as the rightmost one. Then it adds all red components to their attaching paths while maintaining switch-regularity.

More in details, the embedding algorithm works in three phases:

Phase 1: Compute an upward planar embedding of the blue subtree T_b of $RB(T, v)$

Phase 2: Add the weakly regular red components.

Phase 3: Add the strongly regular red components.

We prove that the computed embedding is switch regular after each phase (Lemma 8, Lemma 9, Lemma 10). We start by proving two lemmas that will be useful in order to simplify the description of the algorithm. In Lemma 6, we prove that $RB(T, v)$ has at most two weakly regular red components. Moreover, we prove that if $RB(T, v)$ has two distinct attaching paths, then an attaching path can not have two weak-regular vertices. In Lemma 7, we prove that a weakly regular red component always admits a switch-regular upward planar embedding.

Lemma 6 *Let T be a directed tree and let v be a vertex of T with $deg(v) \geq 3$ and such that $RB(T, v)$ is regular. If $RB(T, v)$ has only one attaching path, then it has at most two weak-regular vertices. If $RB(T, v)$ has two distinct attaching paths, then each of them can have at most one weak-regular vertex. Moreover, there cannot be a weak-regular vertex shared by two distinct attaching paths.*

Proof: First, suppose that there is one attaching path Π and assume by contradiction that it contains the attaching vertices of at least three weakly regular red components. If the three attaching vertices are distinct, then there would be three weak-regular vertices on Π, i.e., a forbidden configuration of type **FC1**. If two of the attaching vertices coincide, then, there is a weak-regular vertex on Π followed by a 2-weak-regular vertex, or viceversa; this implies a forbidden configuration of type **FC2** or **FC3**. If the three attaching vertices all coincide, then there is a 3-weak-regular vertex, i.e., a forbidden configuration of type **FC4**. Consider now the case when $RB(T, v)$ has two distinct attaching paths Π_1 and Π_2 and assume that one of them, say Π_1, has two weak-regular vertices. If Π_1 and Π_2 are equally oriented, then walking along Π_1 starting from v we encounter a branch attaching vertex (possibly v itself) followed by two weak-regular vertices or a 2-weak-regular vertex, i.e. a forbidden configuration of type **FC1** or **FC2**. If Π_1 is incoming and Π_2 is outgoing, then walking along Π_1 starting from v we encounter v which is internal attaching followed by two weak-regular vertices or a 2-weak-regular vertex, i.e., a forbidden configuration of type **FC1** or **FC2**. Finally, consider the case when Π_1 and Π_2 share a subpath and the weak-regular vertex is on this subpath. The last vertex shared by Π_1 and Π_2 is branch attaching and there must be an attaching vertex in the portion of Π_1 not shared with Π_2 because Π_1 and Π_2 are distinct. Thus, walking along Π_1 starting from v we encounter a weak-regular vertex, followed by a branch attaching vertex followed by a regular vertex, i.e., a forbidden configuration of type **FC1**, which is impossible. □

Let s be an angle of a tree T, $\mathsf{prev}(s)$ and $\mathsf{next}(s)$ denote the switches that precede and follow s in the counterclockwise order around T, respectively.

Lemma 7 *Let T be a directed tree with a vertex v, such that $deg(v) \geq 3$. A weakly regular red component C of $RB(T, v)$ admits a switch-regular upward planar embedding.*

Proof: Let Π be the backbone of C, and let $T_1, T_2, \ldots, T_k$ be the hourglass trees obtained by removing the edges of Π in the order their centers are encountered walking along the backbone of C starting from w. Let c_i be the center of T_i, where $i = 1, 2, \ldots, k$. Refer to Figure 14(a) for an illustration of these definitions. Without loss of generality, we may assume that none of $c_1, c_2, \cdots, c_k$ is an endvertex of Π because otherwise Π can be extended up to a leaf of an hourglass tree. We denote by W_0 the backbone Π of C and by W_i the subtree $\Pi \cup T_1 \cup T_2 \cdots \cup T_i$, where $1 \leq i \leq k$.

The switch-regular upward planar embedding of C is constructed in $k+1$ steps. At Step 0, a switch-regular upward planar embedding of Π is chosen. At Step $i > 0$, a switch-regular upward planar embedding of W_i is computed by adding T_i to the switch-regular upward planar embedding of W_{i-1}. We let σ_i denote the counterclockwise sequence of switches in the upward planar embedding of W_i. Each vertex c_j with $j > i$ has only two incident edges in W_i, we denote by $e_{1,j}$ the edge incident to c_j that is traversed first when walking along Π starting from the attaching vertex of C and by $e_{2,j}$ the other one. Let $s_{1,j} = (e_{1,j}, c_j, e_{2,j})$ and $s_{2,j} = (e_{2,j}, c_j, e_{1,j})$ be the two angles at c_j in W_i. We assume that the following invariant holds at Step $i \geq 0$ for vertices c_j with $j > i$: if c_j is neither a source nor a sink, at least one of two pairs $\langle \mathsf{prev}(s_{1,j}), \mathsf{prev}(s_{2,j}) \rangle$ and $\langle \mathsf{next}(s_{1,j}), \mathsf{next}(s_{2,j}) \rangle$ has both switches labeled L. This invariant holds for Step 0 because we choose an upward planar embedding of Π that is switch-regular. We now describe how to add tree T_i at Step i. In the description we denote with σ_{T_i} the counterclockwise sequence of switches in the upward planar embedding of T_i, we denote with $s'_{1,i} = \mathsf{prev}(s_{1,i})$ and $s''_{1,i} = \mathsf{next}(s_{1,i})$, and we denote with $s'_{2,i} = \mathsf{prev}(s_{2,i})$ and $s''_{2,i} = \mathsf{next}(s_{2,i})$. We distinguish the following two cases:

Edges $e_{1,i}$ and $e_{2,i}$ are equally oriented. We add T_i to W_{i-1} such that edges $e_{1,i}$ and $e_{2,i}$ separate the edges entering c_i from those leaving c_i. We prove now that the computed upward planar embedding of W_i is switch-regular. Notice that the upward planar embedding of T_i is switch-regular by Property 2. Since $e_{1,i}$ and $e_{2,i}$ are equally oriented, one of the two switches $s_{1,i}$ and $s_{2,i}$ of W_{i-1} is labeled S and the other one is labeled L. We assume that both $e_{1,i}$ and $e_{2,i}$ enter c_i, the case when they leave c_i is analogous. In this case $s_{1,i} = (e_{1,i}, c_i, e_{2,i})$ is labeled S and $s_{2,i} = (e_{2,i}, c_i, e_{1,i})$ is labeled L. The addition of T_i to c_i will create two new switches, we denote these switches by s_a and s_b (see Figures 14(b) and 14(c)).

The subsequence $s'_{1,i}s_{1,i}s''_{1,i}$ of σ_{i-1} is replaced in σ_i by a sequence $s'_{1,i}s_a\sigma's_b$ $s''_{1,i}$ where $\sigma' \subset \sigma_{T_i}$. The switches s_a and s_b are labeled S and the first and the last switch in σ' are labeled L. Also, since the embedding of T_i is switch-regular there are no two consecutive switches in σ' labeled S. Hence the only possible pairs of consecutive switches labeled S are $s'_{1,i}s_a$ or $s_bs''_{1,i}$. However, this only possible if the sequence $s'_{1,i}s_{1,i}s''_{1,i}$ has a pair of consecutive switches labeled S, which is impossible because the upward planar embedding of W_{i-1} is switch-regular. The subsequence $s'_{2,i}s_{2,i}s''_{2,i}$ of σ_{i-1} is replaced in σ_i by a sequence $s'_{2,i}\sigma''s''_{2,i}$ where $\sigma'' \subset \sigma_{T_i}$. The first

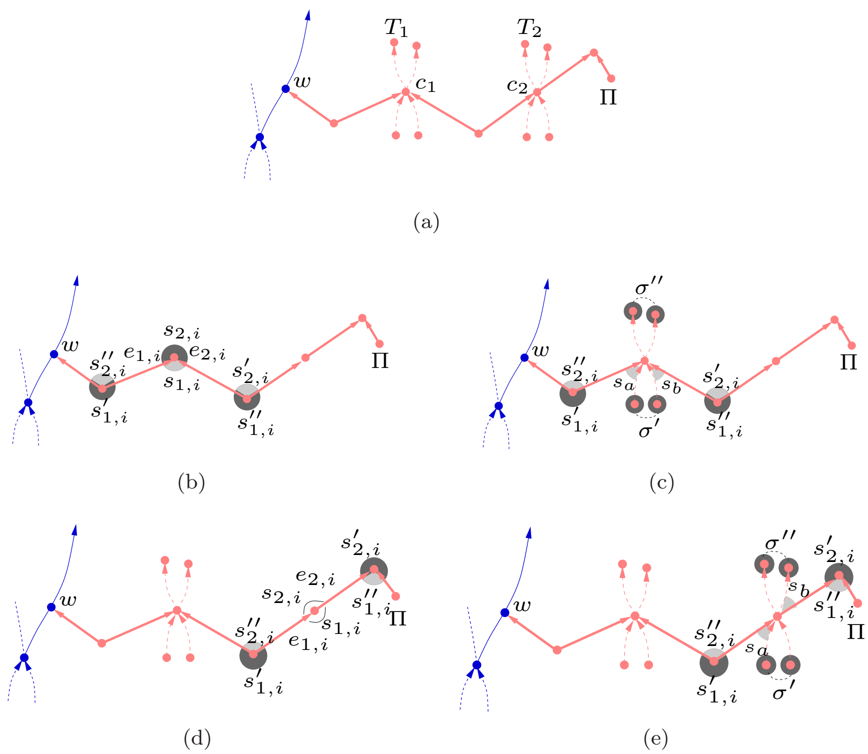


Figure 14: (a) A weakly red component with backbone Π. T_1 and T_2 are two hourglass trees with centers c_1 and c_2, respectively. Dark angles are switches labeled L, light angles are switches labeled S. Figures (b) and (c) (resp. (d) and (e)) show the sequences of the switches before and after the insertion of T_1 (resp. T_2). As shown in the figures, adding T_i ($i \in \{1, 2\}$) does not create any pair of consecutive switches labeled S. In (b) and (c) edges $e_{1,i}$ and $e_{2,i}$ ($i = 1$) are both entering c_i, while in (d) and (e) edge $e_{1,i}$ enters c_i and edge $e_{2,i}$ leaves c_i ($i = 2$).

and the last switch in σ'' are labeled L and there are no two consecutive switches in σ'' labeled S. Hence, there is not a pair of consecutive switches labeled S in the subsequence $s'_{2,i}\sigma'' s''_{2,i}$.

We prove now that the invariant holds for W_i. First observe that we only need to consider the vertices c_j ($j > i$) while walking counterclockwise around W_i, between c_i and $\mathsf{next}(s_{1,i})$. Namely, these vertices are the only ones for which the invariant might not be true due to the addition of T_i. Consider a vertex c_j ($j > i$). We have that $\mathsf{next}(s_{1,j})$ in W_i coincides with $\mathsf{next}(s_{1,i})$ in W_{i-1} and $\mathsf{next}(s_{2,j})$ in W_i coincides with the first switch of σ''. The switch $\mathsf{next}(s_{1,i})$ in W_{i-1} is labeled L, because otherwise $s_{1,i}$ and $\mathsf{next}(s_{1,i})$ would be a pair of consecutive switches labeled S. Also, the first switch of σ'' is labeled L in the upward planar embedding of W_i. It follows that the invariant holds for c_j in W_i.

Edge $e_{1,i}$ enters c_i and edge $e_{2,i}$ leaves c_i. In this case none of the two angles $s_{1,i}$ and $s_{2,i}$ is a switch. By the invariant one of the two pairs $\langle \mathsf{prev}(s_{1,i}), \mathsf{prev}(s_{2,i})\rangle$ and $\langle \mathsf{next}(s_{1,i}), \mathsf{next}(s_{2,i})\rangle$ has both switches labeled L in W_{i-1}. If $\mathsf{prev}(s_{1,i})$ and $\mathsf{prev}(s_{2,i})$ are labeled L, we add T_i to W_{i-1} in such a way that $e_{1,i}$ is the first incoming edge in the counterclockwise order around c_i and $e_{2,i}$ is the first outgoing edge in the counterclockwise order around c_i (see Figures 14(d) and 14(e)). If $\mathsf{next}(s_{1,i})$ and $\mathsf{next}(s_{2,i})$ are labeled L, we add T_i to W_{i-1} in such a way that $e_{1,i}$ is the last incoming edge in the counterclockwise order around c_i and $e_{2,i}$ is the last outgoing edge in the counterclockwise order around c_i. We prove now that the computed upward planar embedding of W_i is switch-regular. Notice that the upward planar embedding of T_i is switch-regular by Property 2.

Assume that $\mathsf{prev}(s_{1,i})$ and $\mathsf{prev}(s_{2,i})$ are labeled L, the other case can be proved symmetrically. The subsequences $s'_{1,i}s''_{1,i}$ and $s'_{2,i}s''_{2,i}$ of σ_{i-1} are replaced in σ_i by a sequence $s'_{1,i}s_a\sigma' s''_{1,i}$ and $s'_{2,i}s_b\sigma'' s''_{2,i}$ where $\sigma', \sigma'' \subset \sigma_{T_i}$. It is easy to see that s_a and s_b are labeled S and that the first and the last switch in σ' and in σ'' are labeled L. Also, since the embedding of T_i is switch-regular there are no two consecutive switches in σ' and in σ'' labeled S. Hence the only possible pairs of consecutive switches labeled S are $s'_{1,i}s_a$ or $s'_{2,i}s_b$. Since $s'_{1,i}$ and $s'_{2,i}$ are labeled L, this case is impossible.

We prove now that the invariant holds for W_i. We still assume that $\mathsf{prev}(s_{1,i})$ and $\mathsf{prev}(s_{2,i})$ are labeled L, the other case can be proved symmetrically. The only vertices c_j with $j > i$ that we must consider are those that, walking counterclockwise around W_i, are encountered between c_i and $\mathsf{next}(s_{1,i})$.

Consider a vertex c_j with $j > i$. We have that $\mathsf{prev}(s_{1,j})$ in W_i coincides with the last switch of σ' and $\mathsf{prev}(s_{2,j})$ in W_i coincides with $\mathsf{prev}(s_{2,i})$ in W_{i-1}. The switch $\mathsf{prev}(s_{2,i})$ in W_{i-1} is labeled L. Also, the last switch of σ' is labeled L in the upward planar embedding of W_i. It follows that the invariant holds for c_j in W_i.

Edge $e_{1,i}$ leaves c_i and edge $e_{2,i}$ enters c_i. This case is symmetric to the previous one.

□

We now describe how to construct a switch-regular upward planar embedding of T. We construct such an embedding by starting from an upward planar embedding of the blue subtree T_b of $RB(T, v)$ and then inserting the regular red components in such a way that the resulting embedding is also switch-regular. The description of the algorithm is simplified by adding a dummy edge and a dummy vertex at the end of each attaching path to guarantee that the endvertex of each attaching path is not an attaching vertex.

Phase 1: Computing an upward planar embedding of T_b.

T_b is an hourglass tree, hence every upward planar embedding of T_b is switch-regular because of Property 2. We choose an embedding of T_b that allows red components to be added while maintaining switch-regularity. Refer to Figure 15. Let Π be an outgoing (incoming) attaching path of $RB(T, v)$. Let u_1, w, u_2 be three vertices encountered consecutively in this order when walking along Π starting from v. We say that Π is *left externally embedded* if: (i) the edge e of Π incident to v is the last outgoing (incoming) edge in the counterclockwise order around v; (ii) for every pair of edges $e_1 = (u_1, w)$ and $e_2 = (w, u_2)$ on Π, the triplet (e_2, w, e_1) is an angle of T_b. Angle (e_2, w, e_1) is said to be the *external angle* of w. We say that Π is *right externally embedded* if: (i) the edge e of Π incident to v is the first outgoing (incoming) edge in the counterclockwise order around v; (ii) for every pair of consecutive edges $e_1 = (u_1, w)$ and $e_2 = (w, u_2)$ on Π, the triplet (e_1, w, e_2) is an angle of T_b. Angle (e_1, w, e_2) is said to be the *external angle* of w. If Π is left (right) externally embedded we say that the external angle of w is a *left* (*right*) angle. We say that Π is *externally embedded* if it is either left externally embedded or right externally embedded. When an attaching path Π is externally embedded, each attaching vertex of Π has at least one external angle. An attaching vertex w may have both left angle and a right angle if all paths leaving (entering) v have a common subpath Π_s and $w \in \Pi_s$.

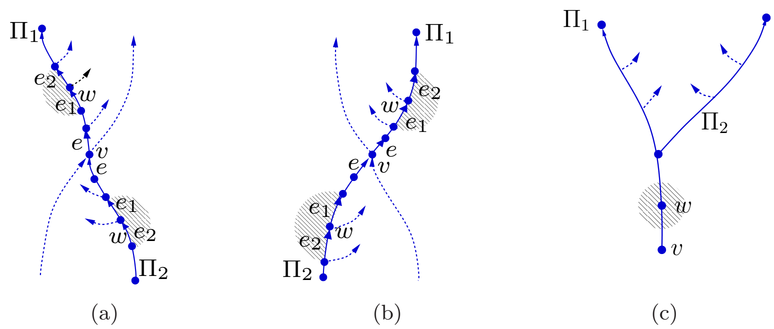


Figure 15: Embedding of the attaching paths. Shaded regions show external angles. (a) Π_1 (Π_2) is an outgoing (incoming) attaching path and it is left externally embedded. (b) Π_1 (Π_2) is an outgoing (incoming) attaching path and it is right externally embedded. (c) Π_1 and Π_2 are both outgoing and share a subpath: w has two external angles.

We describe how to construct an embedding of T_b by distinguishing the following cases:

Case 1: $RB(T, v)$ **has one attaching path.** We choose an upward planar embedding of T_b such that the attaching path is externally embedded.

Case 2: $RB(T, v)$ **has two distinct equally oriented attaching paths.** We choose an upward planar embedding of T_b, such that both attaching paths are externally embedded.

Case 3: $RB(T, v)$ **has one incoming and one outgoing attaching path.** We choose an upward planar embedding, such that both attaching paths are either right externally embedded or both are left externally embedded.

Case 4: $RB(T, v)$ **has two distinct attaching paths that share a subpath.** The two attaching paths Π_1 and Π_2 are equally oriented. Assume they are outgoing, the other case is analogous. Since Π_1 and Π_2 share a subpath, there is a branch attaching vertex which is the last vertex shared by Π_1 and Π_2. Also, there must be an attaching vertex in the portion of Π_2 not shared with Π_1 because Π_1 and Π_2 are distinct. If there is a branch vertex in the subpath shared by Π_1 and Π_2, then $RB(T, v)$ has a forbidden configuration of type **FC1** which is impossible. It follows that, for each vertex u shared by Π_1 and Π_2 that is not branch attaching there is no outgoing incident edge except the one belonging to Π_1 and Π_2. Therefore it is possible, also in this case, to choose an upward planar embedding such that the two attaching paths are externally embedded.

Lemma 8 *Let T be a directed tree with a vertex v, such that $deg(v) \geq 3$. Moreover, let $RB(T, v)$ be regular. Let T_b be the blue subtree of T. The upward planar embedding of T_b computed by Phase 1 is switch-regular and for each external angle s, $\mathsf{prev}(s)$ and $\mathsf{next}(s)$ are labeled L.*

Proof: Since T_b is an hourglass tree, every upward planar embedding of T_b is switch-regular by Property 2. Let Π be the attaching path containing s and assume that Π is an outgoing attaching path, the other case is analogous. If s is a left angle, $\mathsf{prev}(s)$ is the switch at the leaf of the attaching path Π, which is a sink of T_b. Switch $\mathsf{next}(s)$ is either v (if there are no paths entering v) or a leaf of T_b that is a source of T_b. It follows that $\mathsf{prev}(s)$ and $\mathsf{next}(s)$ are both labeled L. □

Phase 2: Adding the weakly regular red components.

We know from Lemma 6 there are at most two weakly regular red components. We assume that there is at least one weakly regular red component because otherwise this phase is not executed. We know from Lemma 7 that each weakly regular red component C has a switch-regular upward planar embedding. Let s^* be the only switch of C at the attaching vertex of C. A switch-regular upward planar embedding of C is a *left embedding* if $\mathsf{prev}(s^*)$ is labeled L, and a *right embedding* otherwise. Note that, in a right embedding $\mathsf{next}(s^*)$ is labeled L (see Figure 16). Given any switch-regular upward planar embedding of C, it is possible to make this embedding a left embedding or a right embedding. When we add a weakly regular red component to T_b we say that C is *left (right) embedded* to mean that C is added inside the left (right) angle of its attaching vertex and a left (right) embedding is chosen for C.

The weakly regular red components are added to the embedding of T_b as follows. If the attaching vertex w of C has only one external angle s, then C will be left or right embedded depending on the fact that s is a left or a right

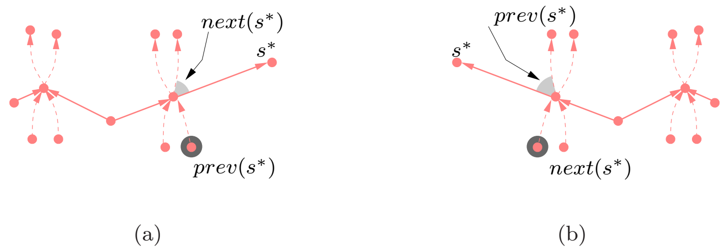


Figure 16: Two switch-regular upward planar embeddings of a weakly regular red component. Dark angles are switches labeled L, light angles are switches labeled S. (a) A left embedding: $\mathsf{prev}(s^*)$ is labeled L. (b) A right embedding: $\mathsf{next}(s^*)$ is labeled S.

angle. If w has two external angles, C is left or right embedded according to the following cases. Refer to Figure 17 for an illustration of these cases.

There is only one attaching path Π. Assume that Π is left externally embedded, the other case is symmetric. If there is only one weakly regular red component, then it is right embedded. If there are two weakly regular red components attached to distinct attaching vertices, then the first attaching vertex w on Π has two external angles (otherwise there is a forbidden configuration of type **FC1**), while attaching vertex w' has either one or two external angles. If w' has only one external angle then this is a left angle because Π is left externally embedded. The weakly regular red component attached to w will be right embedded, the other one will be left embedded. If $w = w'$, then w has two external angles; in this case one of the two components is left embedded and the other is right embedded.

There are one incoming and one outgoing attaching paths. Assume the two attaching paths are left externally embedded, the other case is symmetric. If there is only one weakly regular red component, then it is left embedded. If there are two weakly regular red components, then they are attached to distinct attaching paths by Lemma 6. By hypotheses, one of the two attaching vertices has two external angles, while the other one, call it w', can have one or two external angles. If w' has only one external angle, then this is a left angle because the two attaching paths are left externally embedded. In both cases it is possible to guarantee that both weakly regular red components will be left embedded.

There are two equally oriented attaching paths. An attaching vertex w with two external angles can exist in this case only if the two attaching paths Π_1 and Π_2 share a subpath and w belongs to this subpath. However, by Lemma 6, w cannot be the attaching vertex of a weakly regular red component.

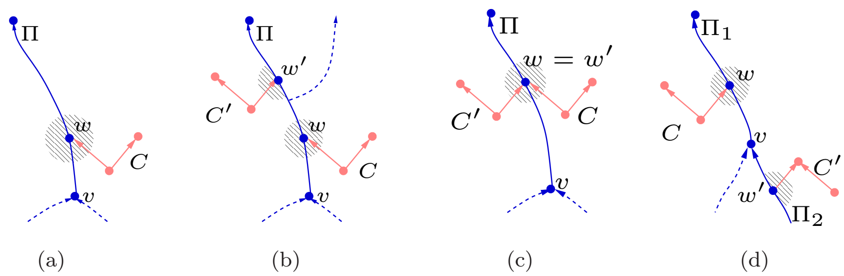


Figure 17: The figure shows how a weakly regular red component whose attaching vertex w has two external angles is added to the embedding of T_b. In (a), (b) and (c) Π is the only attaching path: In (a) there is only one weakly regular red component C, which is right embedded. In (b) and (c) there are two weakly regular red components C and C'. Namely, in (b) $w \neq w'$, then C is right embedded and C' is left embedded; in (c) $w = w'$, then it suffices that one red component is right embedded and the other is left embedded. In (d) there are two attaching paths Π_1 and Π_2 and two weakly red components. In this case both C and C' are left embedded.

Lemma 9 *Let T be a directed tree with a vertex v, such that $deg(v) \geq 3$. Moreover, let $RB(T, v)$ be regular. Let T_W be the subtree of T induced by T_b and the weakly regular red components of $RB(T, v)$. The upward planar embedding of T_W computed by Phase 2 is switch-regular.*

Proof: We consider the following cases:

T has 1 weakly regular red component C. Let Π be the attaching path of $RB(T, v)$ containing the attaching vertex of C and let s be the angle such that C has been added inside s (refer to Figure 18(a)). Assume that s is a left angle and that Π is an outgoing attaching path, the other cases are analogous. Let $s' = \mathsf{prev}(s)$ and $s'' = \mathsf{next}(s)$ in the upward planar embedding of T_b. By Lemma 8, s' and s'' are all labeled L in the upward planar embedding of T_b. Let σ_{T_b}, σ_C, and σ be the counterclockwise sequence of switches in the upward planar embedding of T_b, C, and $T_b \cup C$, respectively. We have $\sigma_{T_b} = s's''\sigma_x$ where σ_x can be empty. After the addition of C we have $\sigma = s'\sigma' s_a s''\sigma_x$, where $\sigma' \subset \sigma_C$. It is easy to see that s_a is labeled S. Also, $\sigma' = \sigma_C \setminus \{s^*\}$, where s^* is the only switch of C at the attaching vertex of C. Since the embedding of C is a left embedding we have that $\mathsf{prev}(s^*)$ in σ_C is labeled L and therefore the last switch in σ' is labeled L. From the switch-regularity of the embedding of C, there is no pair of consecutive switches labeled S in σ'. It follows that the computed upward planar embedding is switch-regular.

T has 2 weakly regular red components C_1 and C_2. Let w_1 and w_2 be the attaching vertices of C_1 and C_2, respectively and let s_1 and s_2 be the two external angles of w_1 and w_2 such that C_1 and C_2 has been added inside s_1 and s_2, respectively. Let $s'_1 = \mathsf{prev}(s_1)$, $s''_1 = \mathsf{next}(s_1)$, $s'_2 = \mathsf{prev}(s_2)$,

$s''_2 = \mathsf{next}(s_2)$ in the upward planar embedding of T_b. By Lemma 8 s'_1, s''_1, s'_2, and s''_2 are labeled L in the upward planar embedding of T_b. If there is only one attaching path or two equally oriented attaching paths, then one between s_1 and s_2 is a left angle and the other one is a right angle; if otherwise the two attaching paths are one incoming and one outgoing, then s_1 and s_2 are both left angles or both right angles. It follows that if we walk counterclockwise on the boundary of T_b, the two edges that define s_1 are traversed opposite to their orientation, while the two edges that define s_2 are traversed coherently to their orientation, or viceversa. Therefore at least one switch is encountered when going counterclockwise from s_1 to s_2 and at lest one switch is encountered when going counterclockwise from s_2 to s_1. This implies that $s'_1 \neq s'_2$ and $s''_1 \neq s''_2$.

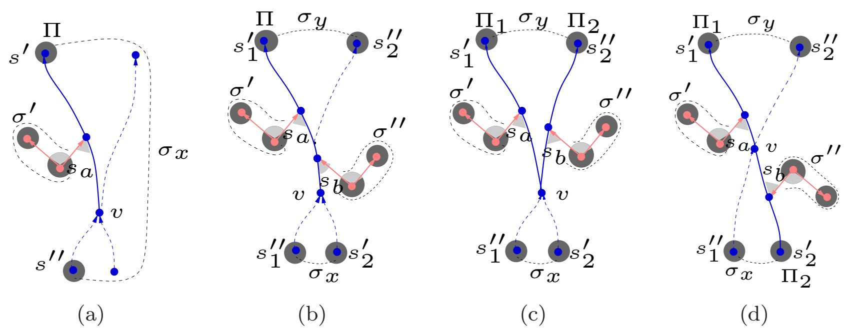


Figure 18: The Figure shows the sequence of the switches after adding the weakly regular red components. Dark angles are switches labeled L, light angles are switches labeled S. In (a) T has 1 weakly regular red component that is left embedded: Its insertion does not create a pair of consecutive S labels. In (b), (c) and (d) T has 2 weakly regular red components. In (b) and in (c) one weakly red component is left embedded, while the other is right embedded: In both cases adding the weakly red components does not create pairs of consecutive S labels. In (d) both the two weakly red components are left embedded; also in this case adding the weakly red components does not create pairs of consecutive S labels.

Let σ_{T_b}, σ_{C_1}, σ_{C_2}, and σ be the counterclockwise sequence of switches in the upward planar embedding of T_b, C_1, C_2, and $T_b \cup C_1 \cup C_2$, respectively. Denote by s^*_i the only switch of C_i at the attaching vertex of C_i ($i = 1, 2$) and let $\sigma' = \sigma_{C_1} \setminus \{s^*_1\}$ and $\sigma'' = \sigma_{C_2} \setminus \{s^*_2\}$. Since $s'_1 \neq s'_2$ and $s''_1 \neq s''_2$, we have $\sigma_{T_b} = s'_1 s''_1 \sigma_x s'_2 s''_2 \sigma_y$ where σ_x or σ_y can be empty, in which case $s''_1 = s'_2$ or $s'_1 = s''_2$.

Consider first the case when there is only one attaching path or there are two equally oriented attaching paths (refer to Figures 18(b) and 18(c)). After the addition of C_1 and C_2 we have $\sigma = s'_1 \sigma' s_a s''_1 \sigma_x s'_2 s_b \sigma'' s''_2 \sigma_y$. It is easy to see that s_a and s_b are labeled S. Since the embedding of C_1 is a left embedding we have that $\mathsf{prev}(s^*_1)$ in σ_{C_1} is labeled L and therefore the last switch in σ' is labeled L. Analogously, since the embedding of C_2 is a right embedding we have that $\mathsf{next}(s_2^*)$ in σ_{C_2} is labeled L and therefore the first switch in σ'' is labeled L. From the switch-regularity of the embedding of C_1 and C_2, there is no pair of consecutive switches labeled S in σ' and σ''. It follows that the sequence $\sigma = s_1'\sigma' s_a s_1'' \sigma_x s_2' s_b \sigma'' s_2'' \sigma_y$ has no pair of consecutive switches labeled S.

Consider now the case when there are two attaching paths one incoming and the other one outgoing (refer to Figures 18(d)). After the addition of C_1 and C_2 we have $\sigma = s_1'\sigma' s_a s_1'' \sigma_x s_2' \sigma'' s_b s_2'' \sigma_y$. It is easy to see that s_a and s_b are labeled S. Since the embedding of C_1 is a left embedding we have that $\mathsf{prev}(s_1^*)$ in σ_{C_1} is labeled L and therefore the last switch in σ' is labeled L. Analogously, since the embedding of C_2 is a left embedding we have that $\mathsf{prev}(s_2^*)$ in σ_{C_2} is labeled L and therefore the last switch in σ'' is labeled L. From the switch-regularity of the embedding of C_1 and C_2, there is no pair of consecutive switches labeled S in σ' and σ''. It follows that the sequence $\sigma = s_1'\sigma' s_a s_1'' \sigma_x s_2' \sigma'' s_b s_2'' \sigma_y$ has no pair of consecutive switches labeled S.

□

Phase 3: Adding the strongly regular red components.

Let T_W be the tree induced by T_b plus the weakly regular red components. Let $C_1, C_2, \dots, C_k$ be the strongly regular red components, and let w_i be the attaching vertex of C_i, where $i = 1, 2, \dots, k$. Since different strongly regular red components may have the same attaching vertex, some of the vertices denoted as w_i, $1 \leq i \leq k$, may coincide. We denote with W_i the subtree $T_W \cup C_1 \cup C_2, \dots \cup C_i$, where $1 \leq i \leq k$. The switch-regular upward planar embedding of T is constructed in k steps starting from an upward planar embedding of T_W computed according to Phases 1 and 2. We denote T_w as W_0. At Step $i > 0$ a switch-regular upward planar embedding of W_i is computed by adding C_i to the switch-regular upward planar embedding of W_{i-1}. Since C_i is an hourglass, every upward planar embedding of C_i is switch-regular by Property 2. Hence, we choose an arbitrary upward planar embedding for C_i. We denote with σ_i the counterclockwise sequence of switches in the upward planar embedding of W_i, and by σ_{C_j} the counterclockwise sequence of switches in the upward planar embedding of C_j. For each W_i, each w_j, where $i = 0, 1, \dots, k-1$, $j = 1, \dots, k$, and $j > i$, has at least one angle, called *candidate angle*, that is either an external angle (see Figures 19(b) and 19(c)) or it is a switch formed by a blue and a red edge (see Figure 19(a)). We assume that for each W_i and for each w_j, where $i = 0, 1, \dots, k-1$, $j = 1, \dots, k$, and $j > i$, the following invariant holds: the upward planar embedding of W_i is switch-regular and there exists a candidate angle s_j at w_j such that: either s_j is not external or if s_j is a left (right) angle, then $\mathsf{next}(s_j)$ ($\mathsf{prev}(s_j)$) is labeled L. Angle s_j is called the *insertion angle* for w_j. Notice that there can be more insertion angles for w_j in W_i. At step $i > 0$ component C_i is added inside one of the insertion angles of its attaching vertex.

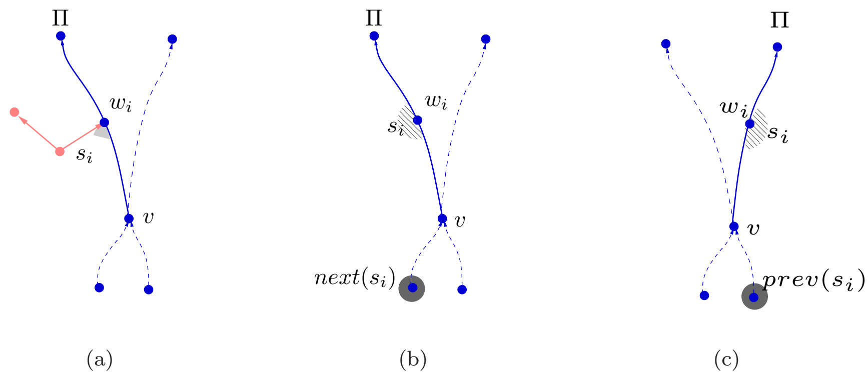


Figure 19: The figure shows the three kinds of candidate angles s_i of the attaching vertex w_i for a strongly regular red component C_i. In (a) s_i is not an external angle. Note that, s_i is labeled S. In (b) s_i is an external left angle: $\mathsf{next}(s_j)$ is labeled L; in (c) s_i is an external right angle: $\mathsf{prev}(s_j)$ is labeled L.

Lemma 10 *Let T be a directed tree with a vertex v, such that $deg(v) \geq 3$. If $RB(T, v)$ is regular, then T is switch-regular.*

Proof: We prove that the upward planar embedding of W_i $(i = 1, \ldots, k)$ is switch-regular. Since $W_k = T$, this implies the statement. The upward planar embedding of $W_0 = T_W$ is switch-regular by Lemmas 8 and 9. We assume now that the invariant holds for W_{i-1} and we prove that the computed upward planar embedding of W_i is switch-regular. Let s_i be the insertion angle of w_i and let $s'_i = \mathsf{prev}(s_i)$ and $s''_i = \mathsf{next}(s_i)$. If s_i is not external, it is a switch labeled S (because w_i is neither a source nor a sink). As a consequence s'_i and s''_i are labeled L otherwise W_{i-1} would not be switch-regular. The subsequence $s'_i s_i s''_i$ of σ_{i-1} is replaced in σ_i by the subsequence $s'_i s_a \sigma' s_b s''_i$, where $\sigma' \subset \sigma_{C_i}$ (see Figure 20(a)). It is easy to see that s_a and s_b are labeled S and that the first and the last switch of σ' are labeled L. From the switch-regularity of the upward planar embedding of C_i there is no pair of consecutive switches labeled S in σ'. Then, there is no pair of consecutive switches labeled S in σ_i.

Assume now that s_i is external. If s_i is a left angle, then the subsequence $s'_i s''_i$ of σ_{i-1} is replaced in σ_i by the subsequence $s'_i \sigma' s_a s''_i$, where $\sigma' \subset \sigma_{C_i}$ (see Figure 20(b)); If s_i is a right angle, then the subsequence $s'_i s''_i$ of σ_{i-1} is replaced in σ_i by $s'_i s_a \sigma' s''_i$, where $\sigma' \subset \sigma_{C_i}$ (see Figure 20(c)). It is easy to see that s_a is labeled S in both cases and that the first and the last switches of σ' are labeled L. If s_i is a left angle then s''_i is labeled L and therefore there is no pair of consecutive switches labeled S in σ_i; If s_i is a right angle then s'_i is labeled L and therefore there is no pair of consecutive switches labeled S in σ_i also in this case.

It remains to prove that the invariant holds for each W_i and for each w_j $(i = 0, 1, \ldots, k-1,\ j = 1, \ldots, k,\ j > i)$. We start showing that the invariant holds for $W_0 = T_W$. W_0 is switch-regular by Lemmas 8 and 9. Let w_j be the attaching vertex of an arbitrary strongly regular red component ($1 \leq j \leq k$) and let Π be the attaching path containing w_j. Vertex w_j has an external angle in W_0 unless it is the attaching vertex of a weakly regular red component. In this case anyway, w_j has a candidate angle that is not external. We prove now that if w_j has only external candidate angles, then the property stated in the invariant holds for one of them.

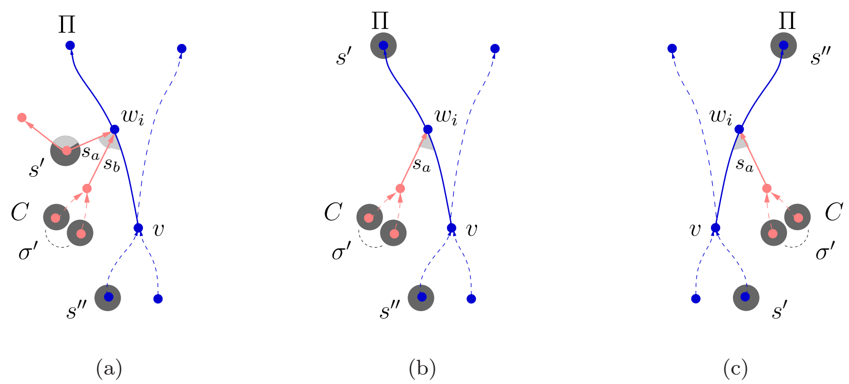


Figure 20: The sequence of switches after the insertion of the strongly regular red component C. Dark angles are switches labeled L, light angles are switches labeled S. In (a) s_i is not an external angle; in (b) s_i is an external left angle; in (c) s_i is an external right angle. In all previous cases there is no pair of consecutive switches labeled S after the insertion of C.

Assume that Π is an outgoing attaching path and that is left embedded. Consider first the case when there is no weakly regular red components attached to Π and vertex w_j has a left angle s_j. Then $\mathsf{next}(s_j)$ is labeled L because it coincides either with v (if there is no path entering v) or with a leaf of T_W which is a source of T_W. Assume now that there is at least one weakly regular red component attached to Π. Let w^* be the attaching vertex of a weakly regular red component that is the farthest from v along Π. We distinguish the following cases:

Vertex w_j is farther from v than w^*. This case can happen only when the following three conditions hold simultaneously: (i) Π is the only attaching path; (ii) w^* has two external angles in T_b; (iii) There is only one weakly regular red component (see Figure 21(a)).

Suppose (i) does not hold; If the two attaching paths are equally oriented, then there exists a branch attaching vertex u (possibly coincident with v). In this case walking along Π starting from v we would encounter either a weak-regular vertex (i.e., w^*) followed by a branch attaching vertex (i.e., u), followed by a regular vertex (i.e., w_j), or a branch attaching vertex (i.e., u) followed by a weak-regular vertex (i.e., w^*) followed by a regular vertex (i.e., w_j), or a vertex that is weak-regular and branch attaching at the same time (in the case when $w^* = u$) followed by a regular vertex (i.e., w_j). In all cases there would be a forbidden configuration of type **FC1** or **FC3**. If one of the attaching paths is incoming and the other one is outgoing, then walking along Π starting from v we would encounter an internal attaching vertex (i.e., v), followed by a weak-regular vertex (i.e., w^*), followed by a regular vertex (i.e., w_j). Also in this case there would be a forbidden configuration of type **FC1**.

If (ii) does not hold, then there must be a branch vertex before w^* along Π. Thus walking along Π starting from v we would encounter a branch vertex followed by a weak-regular vertex (i.e., w^*) followed by a regular vertex (i.e., w_j). This would be a forbidden configuration of type **FC1**.

Finally, if (iii) does not hold, then there is another weak-regular vertex along Π that either precedes w^* (because w^* is the farthest from v) or it coincides with w^*. Then walking along Π starting from v we would encounter two weak-regular vertices followed by a regular vertex (i.e., w_j), or a 2-weak-regular vertex followed by a regular vertex (i.e., w_j). In both cases there would be a forbidden configuration of type **FC1**, or **FC3**.

Since Π is left externally embedded, according to the first case of Phase 2, the weakly regular red component attached to w^* is right embedded. On the other hand w_j has a left angle s_j because Π is left externally embedded. Then $\mathsf{next}(s_j)$ is labeled L because it coincides either with v (if there is no path entering v) or with a leaf of T_W which is a source of T_W.

Vertex w_j is closer to v than w^* or w_j coincides with w^*. If w^* has only one external angle s^* in T_b, then this is a left angle and w_j also has a left angle (see Figure 21(b)). Since w_j is closer to v than w^* or w_j and w^* coincide, $\mathsf{next}(s_j)$ is labeled L because it coincides either with v (if there is no path entering v) or with a leaf of T_W which is a source of T_W. Consider now the case when w^* has two external angles in T_b (see Figure 21(c)). If Π is the only attaching path, then the weakly regular red component attached to Π is either right embedded, if there is only one weakly regular red component, or it is left embedded, if there are two weakly regular red components (in this case the other weakly regular red components is right embedded). In both cases w_j has a left angle s_j because Π is left externally embedded. Since w_j is closer to v than w^* or w_j and w^* coincide, $\mathsf{next}(s_j)$ is labeled L because it coincides either with v (if there is no path entering v) or with a leaf of T_W which is a source of T_W.

This concludes the proof that the invariant holds for W_0. We prove now that the invariant holds for W_i with $i > 0$. We have already proved that W_i is switch-regular. Each attaching vertex w_j distinct from w_i has the same candidate angles that it had in W_{i-1}. For each attaching vertex w_j that coincides with w_i, one of the two angles denoted above as s_a or s_b is the new candidate angle. More precisely, if the candidate angle s_i of w_i in W_{i-1} is external, then s_a is

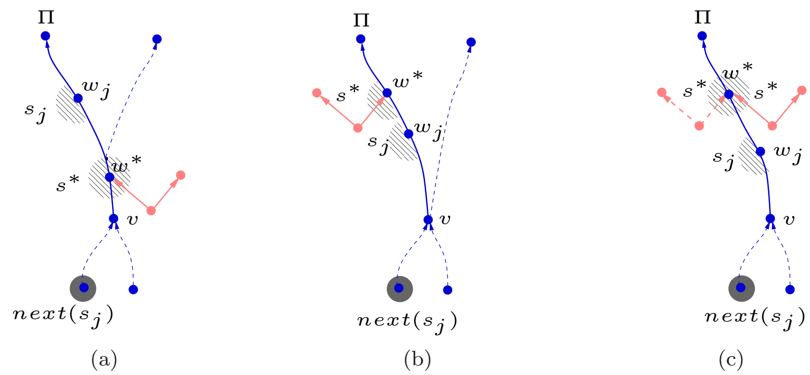


Figure 21: Π is an outgoing attaching path and it is left externally embedded, w^* is the attaching vertex of a weakly regular red component added during Phase 2 and w_j is the attaching vertex of a strongly regular red component that has only external candidate angles s_j. In (a) vertex w_j is farther from v than w^*: The figure shows the only possible configuration that guarantees the absence of forbidden configurations. (b) and (c) refer to the case where either vertex w_j is closer to v than w^* or w_j coincides with w^*. In (b) w^* has only one external left angle, while in (c) w^* has two external angles. In all the cases above $\mathsf{next}(s_j)$ is labeled L.

the new candidate angle at w_i; if s_i is not external, then s_a or s_b is the new candidate angle depending on which has a blue edge (see Figure 20(a)).

We prove now that if w_j has only external candidate angles, then the property stated in the invariant holds for one of them. First of all, notice that the only angles to be considered are the external angles s_j such that $\mathsf{prev}(s_j) = \mathsf{prev}(s_i)$ and $\mathsf{next}(s_j) = \mathsf{next}(s_i)$ in W_{i-1}, because these are the only angles for which the invariant can be lost due to the addition of C_i. Assume that s_i is a left angle, the other case is analogous. Clearly, s_j is also a left angle. If s_j is between $\mathsf{prev}(s_i)$ and s_i when walking counterclockwise around W_{i-1}, then, in W_i, $\mathsf{next}(s_j)$ coincides with the first switch of σ' which is labeled L; If s_j is between s_i and $\mathsf{next}(s_i)$ when walking counterclockwise around W_{i-1}, then in W_i, $\mathsf{next}(s_j)$ coincides with s''_i which is labeled L by induction.

We conclude the proof recalling that at the beginning of the description of the embedding algorithm, in order to simplify the description, we extended the attaching paths with dummy edges. We show now that the removal of these edges does not change the property of switch-regularity of the computed upward planar embedding. Let u^* be the dummy vertex added at the end of an attaching path, and let u be the last "real" vertex of the same attaching path. Clearly, u^* is a leaf and therefore there is only one switch at u^*, which is labeled L. If u is not an attaching vertex, then removing u^*, u becomes a leaf and the only switch at u is labeled L. Hence, in this case the removal of u^* does not change the counterclockwise sequence of switches in the computed upward planar embedding. If u is an attaching vertex, then when we remove

u^*, it becomes either a source or a sink. Thus, one of the switches at u must be labeled L; on the other hand none of the switches existing before the removal of u^* is labeled L. Thus the switch at u labeled L can be only the one created by the removal of u^*. It follows that also in this case the removal of u^* does not change the counterclockwise sequence of switches in the computed upward planar embedding. □

6 Characterization and Test of Switch-regular Upward Planar Trees

In this section, we give three equivalent characterizations of switch-regular trees and present a linear-time algorithm to test if a directed tree is switch-regular.

Theorem 2 *Let T be a directed tree with at least one vertex whose degree is larger than two. The following three statements are equivalent:*

(a). T is switch-regular.

(b). T does not contain 3-hook subdivisions.

(c). There exists a vertex v with $deg(v) \geq 3$ such that $RB(T, v)$ is regular.

Proof: By Lemma 1, (a) implies (b). By Lemma 5, (b) implies (c). Finally, by Lemma 10, (c) implies (a). □

Based on the previous Theorem the following lemma shows that if a directed tree T is switch-regular then $RB(T, u)$ is regular for each vertex u of T with $deg(u) \geq 3$.

Lemma 11 *Let T be a directed tree with at least one vertex whose degree is larger than two and let v_1 and v_2 be any two vertices of T such that $deg(v_1) \geq 3$ and $deg(v_2) \geq 3$. If $RB(T, v_1)$ is regular, then $RB(T, v_2)$ is regular.*

Proof: Since $RB(T, v_1)$ is regular, then by Theorem 2 T does not contain 3-hook subdivisions. By Lemma 5, $RB(T, v)$ is regular for each v with $deg(v) \geq 3$. Hence $RB(T, v_2)$ is regular. □

Corollary 1 *Let T be a directed tree and let v be any (arbitrarily chosen) vertex of T such that $deg(v) \geq 3$. T is switch-regular if and only if $RB(T, v)$ is regular.*

Based on Corollary 1, the testing algorithm arbitrarily chooses a vertex v of T with $deg(v) \geq 3$, and then verifies whether $RB(T, v)$ is regular. $RB(T, v)$ can be easily computed by performing two visits of T starting from v. During the first visit only the edges oriented away from the root are considered, while the second visit considers only the edges oriented towards the root. The vertices reached by one of the two visits are blue vertices, the others are red vertices. Also, the edges traversed during the visits are blue, while the others are red. We also assume that, during the two visits we associate the following information with each vertex u: the colour of u; the number $deg_b(u)$ of blue edges incident to u; the number $deg_r(u)$ of red edges incident to u. The colour of each edge is also stored. Once $RB(T, v)$ is computed the algorithm tests whether $RB(T, v)$ is regular. This is done in three steps by verifying if conditions **RB1**, **RB2**, **RB3** are satisfied. In the description of the algorithms, we will assume that T is rooted at vertex v. As a consequence each red component is rooted at its attaching vertex. Notice that rooting T at v does not imply that all the edges are equally oriented towards or away from the root.

We start with Algorithm 1 to test if condition **RB1** holds. We recall that the last attaching vertex of an attaching path Π is the attaching vertex that has maximum distance from v.

Algorithm 1 TEST-RB1

Input: A red-blue decomposition $RB(T, v)$.
Output: **true** if $RB(T, v)$ satisfies **RB1**, **false** otherwise.
  return TEST-NUMBERATTACHINGPATHS$(v) \leq 2$

Algorithm 2 TEST-NUMBERATTACHINGPATHS

Input: A blue vertex u of $RB(T, v)$.
Output: The number of distinct attaching paths whose last attaching vertex is a descendant of u.
  $index \leftarrow 0$
  $sum \leftarrow 0$
  if $deg_r(u) > 0$ /*u is an attaching vertex*/ **then**
    $index \leftarrow 1$
  end if
  for each blue child w of u **do**
    $sum \leftarrow sum +$TEST-NUMBERATTACHINGPATHS(w)
  end for
  return $\max\{index, sum\}$

Lemma 12 *Let T be a directed tree with n vertices, let v be any (arbitrarily chosen) vertex of T such that $deg(v) \geq 3$. Algorithm* TEST-RB1 *tests whether $RB(T, v)$ satisfies condition **RB1** in $O(n)$ time.*

Proof: First we prove by induction that, given a blue vertex u of $RB(T, v)$, Algorithm 2 correctly computes the number of distinct attaching paths whose last attaching vertex is a descendant of u. If u is a leaf, u is a descendant of itself. Then if u is an attaching vertex the algorithm returns 1, otherwise it returns 0. Suppose now that u is not a leaf. By the inductive hypothesis, if $sum > 0$, then there exist sum distinct attaching paths whose last attaching vertex is a descendant of u. Since in this case u can not be a last attaching vertex, the algorithm returns sum. If $sum = 0$ one of the following two cases occur: if u is an attaching vertex, there exists 1 attaching path that has u as last attaching vertex and then the algorithm returns 1; if u is not an attaching vertex, there does not exist an attaching path that has a descendant of u as last attaching vertex and then the algorithm returns 0.

Since Algorithm 2 is invoked by Algorithm 1 with v as parameter, it computes the number of attaching paths whose last attaching vertex is a descendant of v, namely the number of attaching paths of $RB(T, v)$. Algorithm 1 correctly returns true only if this number is at most two. The time complexity of Algorithm 1 is $O(n)$ since Algorithm 2 performs a preorder visit of the blue vertices of $RB(T, v)$. □

In the description of the following algorithm, used to test condition **RB2**, we assume that when algorithm TEST-RB1 is executed all attaching vertices are stored in a list. Moreover, let u be a vertex of a red component C distinct from the attaching vertex w of C and let u' be the parent of u, we denote by T_u the subtree of C rooted at u and by T'_u the subtree $T_u \cup \{(u, u')\}$.

Algorithm 3 TEST-RB2

Input: A red-blue decomposition $RB(T, v)$.
Output: **true** if $RB(T, v)$ satisfies **RB2**, **false** otherwise.

```
for each attaching vertex w of RB(T, v) do
  for each red component C with attaching vertex w do
    u ← the unique vertex of C adjacent to w
    if TEST-REDCOMPONENT(u, w) > 1 then
      return false
    end if
  end for
end for
return true
```

Algorithm 4 TEST-REDCOMPONENT

Input: A red vertex u of a red component C of $RB(T, v)$, and its parent vertex u'.
Output: 0 if the subtree T'_u of C is a strongly regular red component, 1 if T'_u is a weakly regular red component, $k > 1$ if T'_u is a non-regular red component.

```
index ← 0
sum ← 0
for each child w of u do
  if (u, u') and (u, w) are both leaving or both entering u then
    index ← 1
  end if
  sum ← sum + TEST-REDCOMPONENT(w, u)
end for
return max{index, sum}
```

Lemma 13 *Let T be a directed tree with n vertices, let v be any (arbitrarily chosen) vertex of T such that $deg(v) \geq 3$. Algorithm* Test-RB2 *tests whether $RB(T, v)$ satisfies condition **RB2** in $O(n)$ time.*

Proof: First we prove by induction the correctness of Algorithm 4. Clearly, if u is a leaf, then T'_u is a strongly regular red component and the algorithm returns 0. Suppose now that u is not a leaf. We can have the following cases: $sum = 0$, $sum = 1$, $sum > 1$. Suppose that $sum = 0$; then, by the inductive hypothesis, for each child w_i of u the subtree T'_{w_i} is a strongly regular red component with backbone $\{u\}$. Then if edges (u, w_i) are entering u and edge (u, u') is leaving u (or viceversa), u' is reachable with a directed path from all vertices of T_{w_i} (or all vertices of T_{w_i} are reachable with a directed path from u'), hence T'_u is a strongly regular red component with backbone $\{u'\}$ and the algorithm correctly returns 0. If there exists at least one edge (u, w_i) such that (u, u') and (u, w_i) are both leaving (or both entering) u, then u' is not reachable with a directed path from the vertices of T_{w_i} (or the vertices of T_{w_i} are not reachable with a directed path from u'). In this case the subtree T'_u of C is a weakly regular red component with backbone $\{u', u\}$; Indeed, removing edge (u, u') it remains T_u, which is an hourglass tree with center u, and the algorithm correctly returns 1. Suppose now that $sum = 1$. Then there exists exactly one subtree T'_{w_i} that is a weakly regular red component with backbone $\{u, w_i, \ldots, x\}$. In this case $T_u \setminus T'_{w_i}$ is an hourglass tree with center u and the path $\{u', u, w_i, \ldots, x\}$ is a backbone of T'_u, which is a weakly regular red component. Also in this case the algorithm correctly returns 1. Finally, suppose that $sum > 1$. We have the following two subcases: (i) there exists at least one subtree T'_{w_i} that is not regular. In this case also T'_u is not regular and the algorithm correctly returns a value greater than 1. (ii) There exist at least two subtrees T'_{w_i} and T'_{w_j} that are weakly regular red components with backbones $\{u, w_i, \ldots, x\}$ and $\{u, w_j, \ldots, y\}$, respectively. If T'_u was regular then its backbone should contain both $\{u, w_i, \ldots, x\}$ and $\{u, w_j, \ldots, y\}$ as subpaths, which is clearly impossible. Hence T'_u is not regular and the algorithm correctly returns a value greater than 1.

Since Algorithm 3 invokes Algorithm 4 for each red component of $RB(T, v)$, it returns true if all red components are regular (namely if Algorithm 4 returns a value ≤ 1 for each red component of $RB(T, v)$), false otherwise. About time complexity, Algorithm 4 performs a preorder visit of the vertices of a red component C in $O(|C|)$ time. This implies an $O(n)$ time complexity for Algorithm 3. □

In order to test condition **RB3** we must verify that, for each of the attaching paths none of the forbidden configuration **FC1-FC4** holds (see Algorithm 5). The pseudo-code of sub-routine Test-FC1 is shown in Algorithm 6, the other sub-routines (Test-FC2, Test-FC3, and Test-FC4) are analogous. We assume that the information about whether an attaching vertex is k-weak or k-strong have been associated with each vertex during the execution of Algorithm Test-RB2. We also assume that the (at most) two attaching paths of $RB(T, v)$ have been stored during the execution of Algorithm Test-RB1.

Algorithm 5 TEST-RB3

Input: $RB(T, v)$.

Output: **true** if $RB(T, v)$ satisfies **RB3**, **false** otherwise.

```
Find the unique branch attaching if it exists. Store it in ba
for each attaching path Π_i do
  if (¬TEST-FC1(Π_i, ba)) ∨ (¬TEST-FC2(Π_i, ba)) ∨ (¬TEST-FC3(Π_i, ba)) ∨
  (¬TEST-FC4(Π_i, ba)) then
    return false
  end if
end for
return true
```

Algorithm 6 Test-FC1

Input: An attaching path Π of $RB(T, v)$ and the unique branch attaching vertex of Π, if it exists.

Output: **true** if Π has no forbidden configuration of type **FC1**, **false** otherwise.

```
fc1 ← 0
for each vertex u of Π do
  if fc1 = 0 ∧ ((u is weak) ∨ (u is branch) ∨ (u is internal attaching)) then
    fc1 ← 1
  else if fc1 = 1 ∧ ((u is weak) ∨ (u = ba))) then
    fc1 ← 2
  else if fc1 = 2 ∧ ((u is regular) ∨ (u = ba))) then
    return false
  end if
end for
return true
```

Lemma 14 *Let T be a directed tree with n vertices, let v be any (arbitrarily chosen) vertex of T such that $deg(v) \geq 3$. Algorithm* TEST-RB3 *tests whether $RB(T, v)$ satisfies condition **RB3** in $O(n)$ time.*

Proof: Given an attaching path Π and the unique branch attaching vertex of Π (if it exists), it is immediate to see that Algorithm 6 correctly verifies whether or not Π has a forbidden configuration of type FC1. It is not difficult to see that the other sub-routines TEST-FC2, TEST-FC3, and TEST-FC4 can be written in such a way that they correctly verifies whether or not Π has a forbidden configuration of type FC2, FC3, FC4, respectively. Since these sub-routines are invoked by Algorithm 5 for each attaching path of $RB(T, v)$, then Algorithm 5 correctly returns true if all (at most two) attaching paths of $RB(T, v)$ have no forbidden configurations, false otherwise. About the time complexity, each sub-routine (TEST-FC1, TEST-FC2, TEST-FC3, and TEST-FC4) tests, for each vertex u of an attaching path, whether or not u has some properties (if it is weak, strong, branch, internal attaching or branch attaching).

We prove that each of these properties can be tested in constant time, which implies that both Algorithm 5 and Algorithm 6 have $O(n)$ time complexity. Whether or not u is k-weak, k-strong, and therefore k-regular, can be tested in $O(1)$ time because this information have been stored, for each attaching vertex, during the execution of Algorithm Test-RB2. If $RB(T, v)$ has two attaching paths Π_1 and Π_2 and they are one incoming and the other outgoing then v is internal attaching; if, otherwise, Π_1 and Π_2 are equally oriented we can find in linear time the unique branch attaching vertex by visiting Π_1 and Π_2 from v until we find the last vertex u shared by Π_1 and Π_2 (possibly v itself). To test if a vertex u distinct from v is branch we test if $deg_b(u) \geq 3$. Finally, to test if v is a branch vertex for an incoming (outgoing) attaching path Π we check if v has at least two incoming (outgoing) edges. □

Theorem 3 *Let T be a directed tree with n vertices. There exists an $O(n)$-time algorithm to test whether T is switch-regular and, in the affirmative case, to compute a switch-regular upward planar embedding of T.*

Proof: Given a directed tree T containing a vertex v with $deg(v) \geq 3$, we can test whether T is switch-regular by computing $RB(T, v)$ and then invoking Algorithm 1, Algorithm 3 and Algorithm 5. If Algorithm 1, Algorithm 3 and Algorithm 5 return true, then T is switch-regular. By Corollary 1 and by Lemmas 12, 13 and 14, the time complexity of the testing algorithm is $O(n)$.

If the testing algorithm on T returns true, a switch-regular upward planar embedding of T can be computed according to the techniques described in Section 5. An upward planar embedding of T is completely defined when for each vertex u of T the linear order of the incoming edges of u and the linear order of the outgoing edges of u are specified. Observe that, since both the blue subtree T_b of $RB(T, v)$ and the strong regular red components are hourglass trees, every upward planar embedding of them is switch-regular. Therefore we have to define only the order of the edges incident to the blue vertices of T that belong to an attaching path and the order of the edges incident to the red vertices of T that belong to the backbone of a weakly regular red component. First we describe how to order the edges leaving and entering a vertex that belong to an attaching path of $RB(T, v)$. Let Π be an attaching path of $RB(T, v)$ and let u be a vertex of Π distinct from v. Suppose that Π is an outgoing attaching path, the other case is symmetric. Denote by e_u^+ and e_u^- the two blue edges leaving and entering u that belong to Π, respectively. Observe that all the edges leaving u are blue edges, while e_u^- is the unique blue edge entering u. If Π is left (right) externally embedded we order the edges leaving u such that e_u^+ is the last (first) outgoing edge in the counterclockwise order around u. If u is not an attaching vertex the order of its incidents edges is completely specified, otherwise we have the following cases: (i) u is the attaching vertex of one weakly regular red component and of $k \geq 0$ strongly regular red components. If u has a left (right) external angle, then we order the edges entering u such that the unique red edge of the weakly red component that is incident to u is the first (last) incoming edge and e_u^- is the last (first) incoming edge in the counterclockwise order around u (the others k red incoming edges of u are between these two edges). (ii) u is the attaching vertex of two weakly regular red components and of $k \geq 0$ strongly regular red components (notice that in this case u has two external angles). We order the edges entering u such that the two red edges of the two weakly regular red components incident to u are the first and the last incoming edges in the counterclockwise order around u (e_u^- and the others k red incoming edges of u are between these two edges). (iii) u is the attaching vertex of only strongly regular red components. In this case if u has a left (right) external angle, we order the edges entering u such that e_u^- is the last (first) incoming edge in the counterclockwise order around u. About vertex v, observe that it has only blue incident edges and it belongs to each (at most two) attaching path. Then, if the attaching path is outgoing the edge e_v^- does not exist, analogously if the attaching path is incoming the edge e_v^+ does not exist. If the attaching path Π is outgoing (incoming) and it is left externally embedded, then we order the edges leaving (entering) v such that e_v^+ (e_v^-) is the last outgoing (incoming) edge in the counterclockwise order around v. If Π is outgoing (incoming) and it is right externally embedded, then we order the edges leaving (entering) v such that e_v^+ (e_v^-) is the first outgoing (incoming) edge in the counterclockwise order around v. Finally we describe how to compute a switch-regular embedding for each (at most two) weakly regular red component. Let C be a weakly regular red component with attaching vertex w and let $\Pi = \{w = u_0, u_1, \dots , u_h\}$ be its backbone. To compute a switch-regular embedding of C, for each vertex u_i of Π where $0 < i < h$, we have the following cases: (i) If u_i is a sink-switch (source-switch) of Π we order the edges entering (leaving) u_i such that the two edges of Π entering (leaving) u_i, namely (u_{i-1}, u_i) and (u_i, u_{i+1}), are the first (last) and the last (first) incoming (outgoing) edges in the counterclockwise order around u_i, respectively. (ii) If u_i is not a switch of Π and edge (u_{i-i}, u_i) enters (leaves) u_i, we order the edges entering and leaving u_i such that edge (u_{i-i}, u_i) is the first incoming (last outgoing) edge and edge (u_i, u_{i+i}) is the first outgoing (last incoming) edge in the counterclockwise order around u_i, respectively.

Since ordering the edges incident to a vertex u can be done in $O(deg(u))$ time, then computing a switch-regular embedding of T requires $O(n)$ time. □

An example of a switch-regular embedding of a tree computed by our algorithm is shown in Figure 22.

7 Conclusions and Open Problems

We have addressed a new upward planarity testing problem, that is, the problem of deciding whether an acyclic digraph admits a special kind of upward planar embedding, called a *switch-regular upward planar embedding*. Our research has been motivated by the practical algorithmic impact of this kind of embeddings in several graph drawing applications. We have solved this problem for directed trees, by describing three different characterizations of those digraphs that admit a switch-regular upward planar embedding and a linear-time testing and embedding algorithm. Although directed trees are always upward planar, the

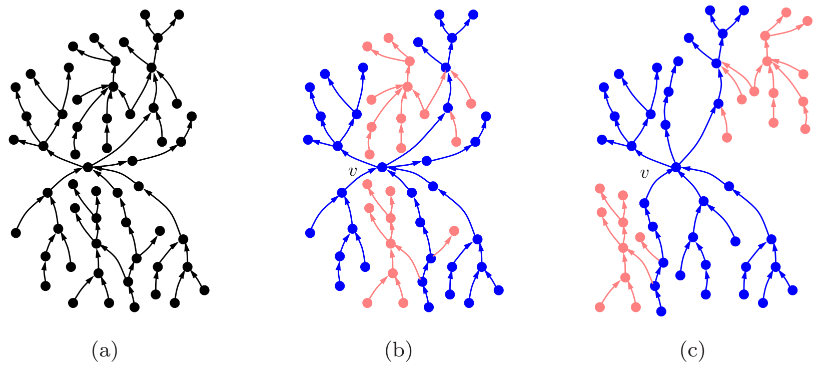


Figure 22: An example of a directed tree and of its corresponding switch-regular embedding computed by our embedding algorithm: (a) A directed tree T; (b) a regular red-blue decomposition of T with respect to v; (c) a switch-regular embedding of T computed by the algorithm using the red-blue decomposition. The algorithm embeds the two attaching paths and the regular red components while maintaining switch-regularity.

work described in this paper shows that the design of an optimal switch-regular upward planarity testing and embedding algorithm for this class of digraphs is not an easy task.

The main open problem on the subject of this paper is that of proving the complexity of testing the existence of switch-regular upward planar embeddings for the case of general acyclic digraphs. Extensions of our results to other subfamilies of planar digraphs are also interesting in our opinion.

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