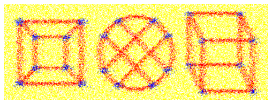
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Clustered Planarity: Small Clusters in Cycles and Eulerian Graphs

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Abstract

We present several polynomial-time algorithms for c-planarity testing for cluster hierarchy $\mathcal{C}$ containing clusters of size at most three. The main result is an $O(|\mathcal{C}|^3 + n)$-time algorithm for clusters of size at most three on a cycle. The result is then generalized to a special class of Eulerian graphs, namely graphs obtained from a 3-connected planar graph of fixed size k by multiplying and then subdividing edges. An $O(3^k \cdot k \cdot n^3)$-time algorithm is presented. We further give an $O(|\mathcal{C}|^2 + n)$-time algorithm for general 3-connected planar graphs.

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1 Introduction

Clustered planarity (or shortly, c-planarity) has recently become an intensively studied topic in the area of graph and network visualization. In many situations one needs to visualize a complicated inner structure of graphs and networks. Clustered graphs—graphs with recursive clustering structures over the vertices—provide a possible model of such a visualization, and as such they find applications in many practical problems, e.g., management information systems, social networks or VLSI design tools [7]. However, from the theoretical point of view, the computational complexity of deciding c-planarity is still an open problem and it is regarded as one of the challenges of the contemporary graph drawing. It was listed as problem no. 35 in [1].

Regarding the graph notations, we follow standard notation on finite loopless graphs. A graph is an ordered pair $G = (V, E)$. By $\overline{G}$ we denote its edge complement (i.e., $(V, \binom{V}{2} \setminus E)$). For a vertex $v \in V$ by $N(v)$ we denote its set of neighbors. For a set $U \subseteq V$, by $G[U]$ we denote the induced subgraph of G with the vertex set U. The number of vertices of the currently discussed graph is denoted by n.

Let $G = (V, E)$ be a graph. Throughout the paper we call $C \subseteq V$ a *cluster*. A *cluster set* on G is a set $\mathcal{C} \subseteq \mathcal{P}(V(G))$ such that for all $C, D \in \mathcal{C}$, either C and D are disjoint or they are in inclusion. A *clustered planar embedding* of $(G, \mathcal{C})$ is a planar embedding emb of G together with a mapping emb_c that assigns to every cluster $C \in \mathcal{C}$ a planar region $emb_c(C)$ whose boundary is a closed Jordan curve and such that

- for each vertex $v \in V$ and every cluster $C \in \mathcal{C}$, it holds that $emb(v) \in emb_c(C)$ if and only if $v \in C$,
- for every two clusters C and D, the regions $emb_c(C)$ and $emb_c(D)$ are disjoint (in inclusion) if and only if C and D are disjoint (in inclusion, respectively), and
- for every edge $e \in E$ and every cluster $C \in \mathcal{C}$, the curve $emb(e)$ crosses the boundary of $emb_c(C)$ at most once.

The pair $(G, \mathcal{C})$ is called *clustered planar* (shortly *c-planar*) if it allows a clustered planar embedding.

It is well known that planar graphs can be recognized in polynomial, even linear time [11]. For c-planarity, determining the time-complexity of the decision problem remains open. It is expected that the problem is hard, but no one managed to prove NP-hardness so far. All what is currently known and published are polynomial-time algorithms for various special cases of the problem. Their significance lies in providing insight into the subtle intricacies of the problem. And even if the general problem turns out NP-hard, they will still provide insight into the boundary between polynomial and hard variants of c-planarity.

One possible approach is to restrict connectivity of subgraphs induced by clusters. For connected clustered graphs (i.e., when all clusters induce connected subgraphs), the problem can be solved in linear time [5]. This work

was extended to "almost" connected clustered graphs in [9, 10] by designing an $O(n^2)$-time algorithm. Another important step was achieved by a characterization of completely connected clustered graphs (where each cluster and its complement induce connected subgraphs): A completely connected clustered graph is c-planar if and only if the underlying graph is planar [2].

Restrictions may also be posed on edges crossing the boundary of clusters. There is an $O(n^3)$-time algorithm for "extrovert" clustered graphs [8], and a linear-time algorithm for clusters with at most four outgoing edges [13].

Several authors consider a fixed embedding of the underlying graph. If the embedding is fixed and each cluster induces at most two connected components, then c-planarity can be tested in polynomial time [12]. A linear-time algorithm for flat cluster hierarchy (i.e., where all clusters are disjoint) in an embedded graph with small faces was found in [6].

Another approach is to impose even more restrictions on the structure of the cluster hierarchy. Polynomial-time algorithms exist for flat cluster hierarchy and the following conditions: the underlying graph is a cycle and clusters are arranged in a cycle [3]; the underlying graph is a cycle and clusters are arranged into an embedded plane graph [4].

We propose to study the situation when all clusters are small. This case has not been considered before and it cannot be solved by explicit application of any existing algorithm. We thus introduce a new class of polynomial c-planarity instances.

So far, we have obtained several results for clusters of size at most three. Our main result is a polynomial-time algorithm for clusters of size at most three on a cycle. This result complements that of [3] where there is a small *number*, namely three, of clusters on a cycle. Similarly to this result, our algorithm is also surprisingly non-trivial. We further generalize our result to a special class of Eulerian graphs that can be obtained from vertex-3-connected planar graphs of fixed size by cloning and subdividing edges.

In Section 2 we remind the notion of saturators and study its meaning in the case of small clusters. We prove that c-planarity of 3-clusters in vertex-3-connected planar graphs is solvable in time $O(|\mathcal{C}|^2 + n)$, where $\mathcal{C}$ is the set of clusters. Section 3 contains the algorithm for 3-clusters on a cycle; it runs in time $O(|\mathcal{C}|^3 + n)$. The generalization to k-Rib-Eulerian graphs running in time $O(3^k \cdot k \cdot n^3)$ is presented in Section 4.

2 Saturators of small clusters

Cortese et al. introduced the following notion in [3]. A set F is a *saturator of* $(G, \mathcal{C})$ if $F \subseteq E(\overline{G})$ and for each cluster $C \in \mathcal{C}$, the vertices of C induce a connected subgraph in $G^F = (V(G), E(G) \cup F)$. The saturator F is called *planar* if G^F is planar. Note that the definition of planar saturator is slightly different here than in [3] (we do not require G^F to be *clustered* planar). The role of saturators is described by the following observation (stated in [3] in an equivalent formulation).

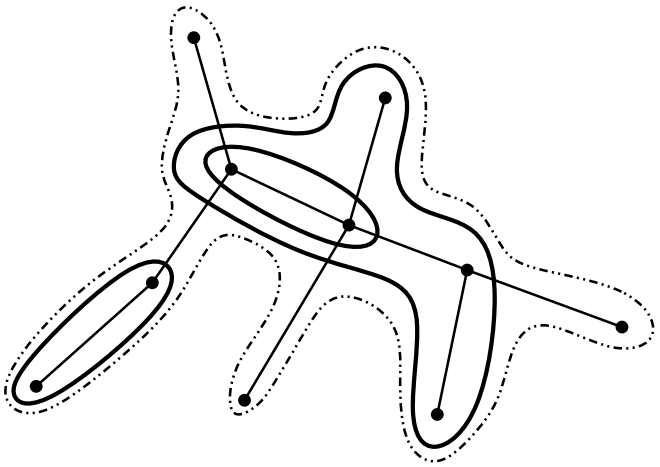


Figure 1: Illustration to the proof of Corollary 1

Lemma 1 *The pair $(G,\mathcal{C})$ is c-planar if and only if there exists a saturator $F = F(G,\mathcal{C})$ such that $(G^F,\mathcal{C})$ is c-planar.*

For a cluster $A \in \mathcal{C}$, we call every pair of its vertices a *candidate edge*. We say that a candidate edge e is *present in a saturator* F if e is an element of F. When it is clear from the context which saturator is considered, we omit its name and speak about *present* candidate edges only.

We further explore the meaning of saturators in certain special cases of graphs G and cluster sets $\mathcal{C}$. In Section 3, we employ this idea in reducing a special case of c-planarity to the existence of a planar saturator of $(G,\mathcal{C})$, and then further to the bipartiteness and triangle-freeness of certain auxiliary graphs.

The following corollary of Lemma 1 is a first step in reducing c-planarity to the existence of a planar saturator. It states that the existence of a planar saturator is sufficient for clusters that do not induce a cycle.

Corollary 1 *The pair $(G,\mathcal{C})$ is c-planar if and only if there exists a saturator F such that $(G^F,\mathcal{C}')$ is c-planar, where $\mathcal{C}' = \{C \in \mathcal{C} : G^F[C]$ contains a cycle$\}$.*

Proof: Given a c-planar embedding of $(G^F,\mathcal{C}')$, insert regions for clusters in $\mathcal{C} \setminus \mathcal{C}'$ inductively from the smallest ones. Each time the boundary of the region $emb(C)$ for $C \in \mathcal{C} \setminus \mathcal{C}'$ is drawn to surround the drawing of $G^F[C]$ so that it contains all the subclusters of C (see Fig. 1). We are surrounding close enough so that $emb(C)$ crosses neither boundaries of other clusters nor undesired edges. □

While Corollary 1 deals with acyclic clusters, we also need to work with clusters that do induce cycles. Then the following theorem [7] is useful.

Theorem 1 *Let $(G, \mathcal{C})$ be a pair whose all clusters induce connected subgraphs. Then $(G, \mathcal{C})$ is c-planar if and only if G is planar and there exists a planar drawing of G such that for each cluster C of $\mathcal{C}$, all the vertices and edges of $G - G[C]$ are in the outer face of the drawing of $G[C]$.*

In this paper we mostly consider clusters of size at most three. We use the fact that if clusters are small, then there are only few possibilities of choosing present candidate edges in a saturator so that each cluster becomes connected.

A highly connected graph imposes other limitations on present candidate edges. Namely, in a fixed planar embedding of a 3-connected planar graph, each candidate edge can be drawn in at most one way. If we restrict ourselves both to 3-connected planar graphs and to clusters of size at most three, then it is possible to test c-planarity effectively.

Theorem 2 *Let G be a 3-connected planar graph and $\mathcal{C}$ a cluster set containing only clusters of size at most three. Then the c-planarity of $(G, \mathcal{C})$ can be decided in time $O(|\mathcal{C}|^2 + n)$.*

Proof: As the graph is 3-connected, each candidate edge e is either an edge of G, or there is at most one face that e can be drawn in. Thus the solution consists only in choosing candidate edges that are present in the sought planar saturator.

We use a reduction to an instance of 2-SAT. For each candidate edge e we introduce a boolean variable x_e saying if e is present (TRUE) or not (FALSE). We create four types of clauses:

- Clauses saying that each cluster is connected. For each three-cluster $\{a_1, a_2, a_3\} \in \mathcal{C}$ we create clauses

  $$(x_{\{a_1,a_2\}} \vee x_{\{a_2,a_3\}}) \wedge (x_{\{a_1,a_2\}} \vee x_{\{a_1,a_3\}}) \wedge (x_{\{a_1,a_3\}} \vee x_{\{a_2,a_3\}}),$$

  and for each two-cluster $\{a_1, a_2\} \in \mathcal{C}$ we create the clause $(x_{\{a_1,a_2\}})$.

- Clauses saying that there is no crossing. For candidate edges $\{a, b\}$ and $\{c, d\}$ whose cyclic order along a common face of G is $acbd$, we create the clause $(\neg x_{\{a,b\}} \vee \neg x_{\{c,d\}})$.

- We use just candidate edges that can be drawn; we create the clause $(\neg x_{\{a,b\}})$ for each candidate edge $\{a, b\}$ such that a and b do not lie in a common face of G.

- For each candidate edge $\{a, b\}$ that is also an edge of G we create the clause $(x_{\{a,b\}})$.

If the formula is not satisfiable, then clearly no saturator exists and $(G, \mathcal{C})$ is not c-planar. Otherwise, let ν be a satisfying valuation and $F = F(\nu)$ the corresponding set of edges. Let $\mathcal{C}'$ be the set of clusters that induce cycles in G^F.

By the definition of F, the graph G^F is planar. Hence, if $\mathcal{C}'$ is empty, then by Corollary 1 the pair $(G, \mathcal{C})$ is c-planar.

Suppose that $\mathcal{C}'$ is nonempty and let $C \in \mathcal{C}'$. Then C induces a triangle T. If at least one edge of T is neither an edge of G nor a two-cluster in $\mathcal{C}$, we may set the variable corresponding to this edge to false, and thus obtain another satisfying valuation ν for which $\mathcal{C}'$ contains less clusters. We do this for every such cluster.

It is easy to see that the remaining clusters in $\mathcal{C}'$ induce triangles in G^F for *any* saturator F. Let F be an arbitrary planar saturator. If there is such a triangle, say T, that is not a boundary of a face of G^F, then both the interior and exterior of T are non-empty. By the 3-connectivity of G and G^F, the same is true for T in any planar drawing of G^F. Then G^F cannot be drawn so that all of $G - G[C]$ is outside $G[C]$, and by Theorem 1, $(G^F, \mathcal{C}')$ is not c-planar. Since we have proved this for an arbitrary planar saturator F, by Corollary 1 the pair $(G, \mathcal{C})$ is not c-planar either.

Otherwise, all triangles induced by clusters of $\mathcal{C}'$ are boundaries of faces of G^F. Since clusters are disjoint, there always exists a face f whose boundary is not fully contained in a cluster. Let f be the outer face; then the drawing fulfills the conditions of Theorem 1 and $(G^F, \mathcal{C}')$ is c-planar, hence $(G, \mathcal{C})$ is c-planar as well.

This algorithm runs in time $O(|C|^2 + n)$. The formula has $O(|C|)$ variables and at most $O(|C|^2)$ clauses. The clauses can be produced by examining the vertices of clusters along the boundary of each face, which can be done in total time $O(n)$, as well as comparing induced triangles with boundaries of faces. The time needed to solve the instance of 2-SAT is $O(|C|^2)$. □

3 Three-clusters on a cycle

In this section we present an algorithm that decides c-planarity for clusters of size at most three on a cycle. In Subsection 3.1 we describe the crucial part of the algorithm—the construction of auxiliary graphs. We then characterize the c-planarity of the input instance by the properties of the auxiliary graphs. This characterization is then proved in Subsection 3.2 and Subsection 3.3, which also contains time analysis of the algorithm.

Definition 1 *Let G be a cycle and $\mathcal{C}$ a set of at most three-element clusters on $V(G)$. We say that two candidate edges* conflict *if the cyclic order of their vertices is abab.*

We say that two three-vertex clusters A and $B \in \mathcal{C}$

- intersect *if the cyclic order of their vertices along G is aabbab,*
- alternate *if the cyclic order of their vertices along G is ababab.*

Given two clusters A and B we say that the vertices $a \in A$ and $b \in B$ are consecutive *if there exists a path in G from a to b that uses no other vertices of A or B.*

Lemma 2 *If G is a cycle and $\mathcal{C}$ contains only clusters of size at most three, then $(G, \mathcal{C})$ is c-planar if and only if there exists a planar saturator F.*

Proof: Note that (regardless of $\mathcal{C}$) any planar drawing of G forms a Jordan circle with all vertices on its boundary.

To prove the *'if'* implication, we take a planar saturator F and use Lemma 1. If the cluster C forms a triangle T in G^F with edges e_1, e_2 and e_3, then two of its edges (without loss of generality e_1 and e_2) must be represented in the same face. Now we remove embedding of e_3 and we redraw it along the union of e_1 and e_2. The cluster containing this triangle is represented by a curve surrounding this triangle (whose interior is empty). The remaining cases are covered by Corollary 1. The *'only if'* implication is a straightforward corollary of the Lemma 1. □

3.1 Construction of auxiliary graphs G_1, G_1^M, and G_2

We are given a pair $(G, \mathcal{C})$, where G is a cycle and $\mathcal{C}$ contains only clusters of size at most three. According to Lemma 2, deciding the c-planarity of $(G, \mathcal{C})$ amounts to finding a planar saturator F. Thus, we need to pick suitable candidate edges for F so that the graph G^F makes every cluster connected and has a planar embedding.

As we want to use the algorithm for finding candidate edges also in more general setting in subsequent sections we design it to take a pseudocluster set as an input. A *pseudocluster set* on G is a set $\mathcal{C} \subseteq \mathcal{P}(V(G))$ such that for all $C, D \in \mathcal{C}$, either C and D are disjoint, they are in inclusion or $|C| = |D| = 2$. Note that the notion of pseudocluster set differs from the notion of cluster set just by allowing intersection of clusters of size two with other clusters of size two.

Since G is a cycle, we only distinguish two ways of drawing a candidate edge in a planar embedding: inside or outside the cycle G.

Conflicts of candidate edges impose restrictions on their embedding. For two-vertex clusters, the situation is evident: each candidate edge must be drawn on one side of the cycle and any conflicting candidate edge must be drawn on the other side.

For three-vertex clusters the situation is more complicated, because we do not know in advance which candidate edges are present in the sought saturator and which are not. However, since F is a saturator, we know that every cluster C is connected in G^F. Hence, out of every pair of candidate edges of C, at least one is present in F. Thus, we consider pairs of three-vertex-cluster candidate edges; these become vertices of an auxiliary graph G_1. Edges of two-vertex clusters will become vertices of G_1 also.

The formal construction of G_1 can be found below. There we also formalize the correspondence between vertices of G_1 and (pairs of) candidate edges. Here, for convenience, we use the notion of correspondence in an intuitive way.

For some vertex pairs x and y of G_1 the following holds: if any candidate edge corresponding to x is present, then any present candidate edge corresponding

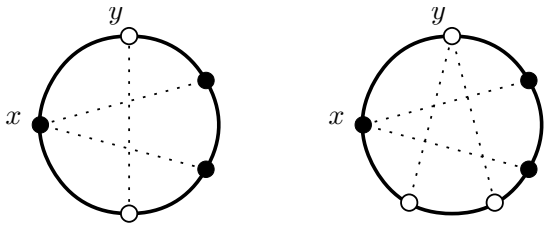


Figure 2: Some of the situations when any candidate edge corresponding to x must be drawn on the other side of the cycle than any candidate edge corresponding to y.

to y must be drawn on the other side of the cycle. Otherwise, a crossing would occur. Figure 2 illustrates some of those cases. We represent such a case by the edge between x and y in G_1.

We observe that if a vertex x is non-isolated in G_1, then all present candidate edges corresponding to x must be drawn on a common side of the cycle. For adjacent vertices the sides are distinct. The idea is to find a bipartition of G_1. If it exists, we may obtain a drawing of all present candidate edges such that all edges corresponding to vertices in one part will be drawn inside the cycle and all edges corresponding to the other part will be drawn outside the cycle. Isolated vertices in G_1 are exceptional and we will not consider them to belong to any bipartity of G_1, because their corresponding candidate edges have, in a sense, more freedom.

The graph G_1 does not capture well the restrictions caused by alternating clusters—there may be several pairwise-alternating clusters that do not give rise to any edge of G_1. Hence, we define another auxiliary graph G_2. The vertices of G_2 are three-vertex clusters, and edges $\{A, B\}$ of G_2 express that clusters A and B alternate. We later prove that, for c-planarity, there may be no triangle in G_2.

In some cases there are vertices of G_1 whose corresponding candidate edges "behave in the same way" in any planar drawing of a saturator: either all present edges are drawn outside, or all inside, or all may be drawn on both sides. In the bipartition language, such vertices must either belong to a common bipartity of G_1 in any bipartition, or they must be all isolated. We need to "unify" them. Hence, we construct the graph G_1^M from G_1 by repeated merging of certain vertex tuples into groups.

The formal construction of the graphs G_1, G_2 and G_1^M follows. It is illustrated in Fig. 3.

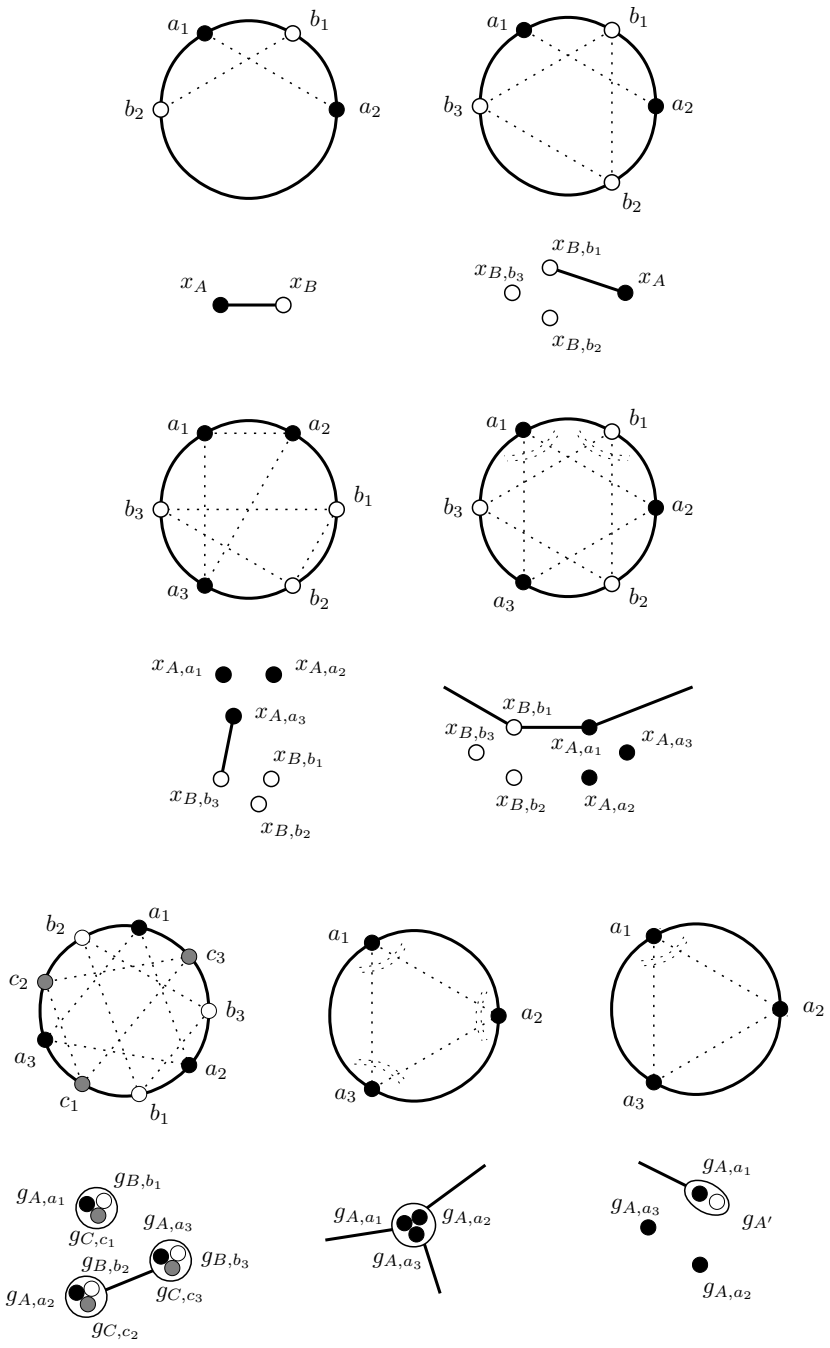


Figure 3: Illustration of rules 1, 2, 3, 4, 5, 6, and 7.

Algorithm: Construction of G_1
Input: $G = (V, E)$, a pseudocluster set $\mathcal{C}$
Output: the graph G_1

$V(G_1) := \{x_{A,v} : A \in \mathcal{C}, |A| = 3, v \in A\} \cup \{x_A : A \in \mathcal{C}, |A| = 2\}$
$E(G_1) := \emptyset$

Rule 1. For every two clusters $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ whose vertices have the cyclic order a_1, b_1, a_2, b_2, set $E(G_1) = E(G_1) \cup \{\{x_A, x_B\}\}$.

Rule 2. For every two clusters $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3\}$ whose vertices have the cyclic order a_1, b_1, a_2, b_2, b_3, set $E(G_1) = E(G_1) \cup \{\{x_A, x_{B,b_1}\}\}$.

Rule 3. For every two clusters $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ whose vertices have the cyclic order $a_1a_2b_1b_2a_3b_3$, set $E(G_1) := E(G_1) \cup \{\{x_{A,a_3}, x_{B,b_3}\}\}$.

Rule 4. For every two alternating clusters A and B such that the vertices y_A and y_B both have degree exactly one in G_2, and for every pair of their vertices $a_i \in A$ and $b_j \in B$ that are consecutive and both non-isolated in G_1, set $E(G_1) := E(G_1) \cup \{\{x_{A,a_i}, x_{B,b_j}\}\}$.

Definition of G_2

$$V(G_2) := \{y_A : A \in \mathcal{C}, |A| = 3\}$$

$$E(G_2) := \{\{y_A, y_B\} : A \text{ and } B \text{ alternate}\}.$$

To formalize the correspondence of clusters and candidate edges with vertices of G_1 and G_2, we introduce the following definition.

Definition 2 *We say that a candidate edge $e = \{a_i, a_j\}$ of a cluster A in G* corresponds *to a vertex v of G_1 if $v = x_{A,a_i}$ or $v = x_{A,a_j}$, or if $v = x_A$.*

We say that a cluster B corresponds *to a vertex u of G_2 if $u = y_B$.*

The following algorithm "Construction of G_1^M" constructs the graph G_1^M from G_1 together with a mapping $g : V(G_1) \to V(G_1^M)$. The vertices of G_1^M will represent groups of vertices of G_1 and the mapping g will assign to each vertex the group to which it belongs. For short, we write g_{A,a_i} or g_A instead of $g(x_{A,a_i})$ or $g(x_A)$, respectively.

The algorithm "Construction of G_1^M" starts with one-vertex groups equal to vertices of G_1 and then merges certain groups using the following procedure.

Procedure: Merge
Input: vertex groups $g_1, g_2, \ldots, g_k \in V(G_1^M)$
Output: modifies G_1^M

1. Replace the groups $g_1, g_2, \ldots, g_k$ with a newly created vertex group w, and set the edges in G_1^M so that
$$N(w) = N(g_1) \cup N(g_2) \cup \cdots \cup N(g_k) \setminus \{g_1, g_2, \ldots, g_k\}.$$
2. If there are two indices $1 \leq i, j \leq k$ (not necessarily distinct) such that g_i and g_j are adjacent then add a loop $\{w, w\}$.
3. For all vertices v in $g^{-1}\big(g_1 \cup g_2 \cup \cdots \cup g_k\big)$ set $g(v) := w$.

Algorithm: Construction of G_1^M
Input: the graph G_1
Output: the graph G_1^M, a mapping $g : V(G_1) \to V(G_1^M)$
$G_1^M := G_1$, $g := \mathrm{id}$

Rule 5. For each cluster $A = \{a_1, a_2, a_3\}$ which alternates with at least two other clusters $B = \{b_1, b_2, b_3\}$ and $C = \{c_1, c_2, c_3\}$ in the way a_1, c_3, b_3, a_2, b_1, c_1, a_3, c_2, b_2, do merge$(g_{A,a_1}, g_{B,b_1}, g_{C,c_1})$, merge$(g_{A,a_2}, g_{B,b_2}, g_{C,c_2})$, and merge$(g_{A,a_3}, g_{B,b_3}, g_{C,c_3})$.

Rule 6. For every three-vertex cluster A having all corresponding vertices g_{A,a_i} non-isolated in G_1^M, do merge$(g_{A,a_1}, g_{A,a_2}, g_{A,a_3})$.

Rule 7. For every two clusters $A' = \{a_1, a_2\}$ and $A = \{a_1, a_2, a_3\}$ such that g_{A,a_1} is not isolated in G_1^M, do merge$(g_{A,a_1}, g_{A'})$.

Having constructed all the auxiliary graphs G_1, G_1^M and G_2, it is easy to decide if the input pair $(G, \mathcal{C})$ is c-planar, as stated in the following theorem. The theorem is proved in subsections 3.2 and 3.3.

Theorem 3 *Let G be a cycle, let $\mathcal{C}$ contain only clusters of size at most three, and let G_1^M and G_2 be the graphs constructed for $(G, \mathcal{C})$ using the algorithms "Construction of G_1", "Construction of G_1^M", and "Definition of G_2". Then the pair $(G, \mathcal{C})$ is c-planar if and only if G_1^M is bipartite and G_2 is triangle-free.*

3.2 The proof of Theorem 3, part 1: The necessary condition

Lemma 3 *Let F be a planar saturator. Then in G^F, the following is true:*

1. *if there is an edge between the vertices x and y in G_1, then any present candidate edge corresponding to x is drawn inside the cycle G, and any*

present candidate edge corresponding to y is drawn outside the cycle G, or vice versa.

2. *the present candidate edges corresponding to vertices in $g^{-1}(v)$ such that v is non-isolated in G_1^M are drawn either all inside or all outside the cycle G.*

3. *if there is an edge between the vertices x and y in G_1^M then any present candidate edge corresponding to $g^{-1}(x)$ is drawn inside the cycle G, and any present candidate edge corresponding to $g^{-1}(y)$ is drawn outside the cycle G, or vice versa.*

Proof: We follow the construction of G_1 and prove inductively that part 1 holds after every step. Before any rule is applied, there are no edges in G_1 and it holds trivially. Then a step according to rule 1, 2, or 3 adds one new edge, say, xy. For all these rules, it is not hard to see that if an edge corresponding to x and an edge corresponding to y are drawn on the same side of the cycle G, then they cross each other. Hence, after an application of rule 1, 2, or 3, part 1 remains valid.

Let us consider a step according to rule 4. We use the same notation as in the description of this step, so we have two clusters A and B, and let $x = x_{A,a_1}$ and $y = x_{B,b_1}$. Note that by definition of rule 4, the vertices x and y are already non-isolated. Thus all present candidate edges corresponding to x must be drawn on the same side of the cycle, by induction hypothesis. The same holds for y.

Assume for contradiction that there are candidate edges corresponding to x and y both inside the cycle G. Then it must be edges a_1a_3 and b_1b_2, because they are the only pair without a crossing. By the above argument, the edge a_1a_2 can only be drawn on the same side as a_1a_3; but that is not possible because of b_1b_2. So a_1a_2 is not present. Then a_2a_3 is present and drawn outside. Similarly, b_1b_3 is not present, and there is no way to draw b_2b_3, a contradiction.

Part 2 is proved by induction on the number of mergings taken by the algorithm "Construction of G_1^M". To simplify the proof, we reorder the mergings done due to rule 5, so that we first merge the groups that are always non-isolated. We claim that after all mergings of rule 5 have been applied, the resulting graph G_1^M depends only on the relative position of the clusters, not on the order of the mergings. Although this is not hard to see directly, we prove this claim formally as follows.

For the purpose of this paragraph, consider an equivalence $\sim$ on vertices of G_1 defined by $u \sim v$ if and only if $g(u) = g(v)$. Observe that if we know G_1 and $\sim$, then G_1^M is fully determined, since g is surjective and $g(u)$ is adjacent to $g(v)$ if and only if there are $u', v' \in V(G_1)$ such that $u' \sim u$, $v' \sim v$ and u' is adjacent to v' in G_1 (this is true at the beginning and the procedure Merge can be easily checked not to break it). Now consider (just for this paragraph) for any set $\{v_1, v_2, \ldots, v_k\}$ of vertices of G_1 a procedure $\overline{\text{Merge}}(v_1, \ldots, v_k)$ that consists only of a single call of $\text{Merge}(g(v_1), \ldots, g(v_2))$. Any call of Merge in the algorithm "Construction of G_1^M" can be easily replaced by a call of $\overline{\text{Merge}}$.

Moreover a list of all $\overline{\text{Merge}}$s to be done according to rule 5 can be made before actually doing any of them, because it only depends on the relative position of the clusters. Finally, observe that the relation $\sim$ is the transitive closure of the relation "being together arguments of a call of $\overline{\text{Merge}}$". Since this closure is unique and the graph G_1^M is determined by $\sim$, the order of the merges does not matter.

Now let us get back to the proof of part 2. Before any merging was done, the claim holds as a consequence of part 1. We prove the validity of the claim after each merging. We distinguish several cases of merging, using the same notation as in the description of the steps:

(a) Merging due to rule 5: We distinguish two subcases of the merging and we reorder the mergings in such a way that all the mergings of type (i) are done before the first one of type (ii).

(i) Merging of g_{A,a_2}, g_{B,b_2} and g_{C,c_2}, or of g_{A,a_3}, g_{B,b_3} and g_{C,c_3}: consider the first merge. If $g^{-1}(g_{A,a_2})$ is different from x_{A,a_2}, then g_{A,a_2} has already been merged with something and it could be only through a merge of type (i) due to the order of merges. Hence g_{A,a_2} is non-isolated and by the induction hypothesis we know that it is drawn on one side. The same holds for g_{B,b_2} and g_{C,c_2}. So it is sufficient to show that no candidate edge corresponding to x_{A,a_2} and x_{C,c_2} is drawn outside if there is a candidate edge corresponding to x_{B,b_2} drawn inside.

Assume that there is candidate edge b_1b_2 drawn inside. If c_1c_2 is outside then there is no way to connect a_3. If a_1a_2 is outside then it is impossible to connect c_3. If c_2c_3 is outside, then we must draw a_2a_3 outside in order to connect a_3, and a_1a_2 inside to connect a_1. But then we have no chance to connect b_3. With a_2a_3 outside we must draw c_2c_3 outside to connect c_3 and the situation is exactly the same as the previous case.

So let us consider b_2b_3 inside. If there is a_1a_2 outside then we can not connect c_3, and the same holds for candidate edge c_2c_3 outside and vertex a_1. If the candidate edge c_1c_2 is outside, then a_1a_2 must be outside to connect a_1, but we already know that this is not possible. Similarly a_2a_3 outside forces us to draw c_2c_3 outside to connect c_3.

To sum it up, if there is a candidate edge corresponding to b_2 inside the cycle G then all the edges corresponding to a_2 and c_2 can only be drawn inside, too. The proof for the second merge is the same one with the role of the letters a and b and indices 2 and 3 swapped.

(ii) Merging of g_{A,a_1}, g_{B,b_1} and g_{C,c_1}: If all g_{A,a_1}, g_{B,b_1} and g_{C,c_1} are isolated in G_1^M then there is nothing to prove. Assume without loss of generality that g_{A,a_1} is non-isolated. We know that if both candidate edges corresponding to x_{A,a_1} are present, then they must be on the same side. Moreover, we know that if any present candidate edge corresponding to x_{A,a_2} is inside, then any present candidate edge

corresponding to x_{A,a_3} is outside and vice versa. Then a_2a_3 can not be present, and the edges a_1a_2 and a_1a_3 are on the opposite sides. But that is not possible by the restriction on x_{A,a_1}, and the instance is not c-planar, a contradiction.

(b) Merging due to rule 6: Let $A = \{a_1, a_2, a_3\}$ be the cluster on which the step is taken. By the induction hypothesis we know that all present candidate edges corresponding to g_{A,a_1}, are drawn on one side of the cycle, because g_{A,a_1} is non-isolated, and the same holds for g_{A,a_2} and g_{A,a_3}. It is sufficient to show that for their representatives $x = x_{A,a_1}$, $y = x_{A,a_2}$ and $z = x_{A,a_3}$, their present candidate edges must be on the same side. Take two candidate edges corresponding to a combination of x, y and z. They meet in one vertex of cluster A, and that forces them to be drawn on the same side.

(c) Merging due to rule 7: By the induction hypothesis we know that all present candidate edges corresponding to the group g_{A,a_1} are drawn on one side of the cycle, because g_{A,a_1} is non-isolated. If $g^{-1}(g_{A'})$ is different from $x_{A'}$ then it was merged in an application of rule 7, so it is non-isolated. This means that the group $g_{A'}$ is drawn on one side of the cycle, too. It is sufficient to show that for their representatives $x = x_{A,a_1}$ and $y = x_{A'}$, their present candidate edges must be on the same side. The candidate edge corresponding to y is one of the two candidate edges corresponding to x. Thus, if all the edges of x are drawn inside, then so is y. Similarly, if x is outside, then y is outside, too.

Part 3 is an easy consequence of the first two and the fact that, if there is an edge between two vertices x and y of G_1^M, then there exists an edge between two vertices u and v of G_1, where $u \in g^{-1}(x)$ and $v \in g^{-1}(y)$. □

Lemma 4 *If there are three clusters $A, B, C \in \mathcal{C}$ such that A alternates with B and B alternates with C, then either A alternates with C or A intersects C.*

Proof: Vertices of cluster B split the cycle into three segments. As B alternates both with A and C, each of these three segments contains exactly one vertex of A and one vertex of C. Therefore there are two possible cyclic orders of vertices of A and C. Either it is *acacac* or it is *acacca*. In the first case, A alternates C, in the second case, A intersects with C. □

Lemma 5 *If G has a planar saturator, then G_1^M is bipartite and G_2 is triangle-free.*

Proof: First assume that G_1^M is not bipartite. Then it contains an odd cycle C, $V(C) = \{v_1, v_2, v_3 ... v_{2k+1}\}$, $0 \leq k$. Without loss of generality we can assume that candidate edges corresponding to vertices in $g^{-1}(v_1)$ are drawn inside the cycle. By Lemma 3 we know that $g^{-1}(v_2)$ is outside, by the same argument $g^{-1}(v_3)$ is inside, etc. But then both $g^{-1}(v_1)$ and $g^{-1}(v_{2k+1})$ are drawn inside, which is not possible again by Lemma 3, a contradiction.

If G_2 contains a triangle, then there are three pairwise alternating clusters, say A, B, and C. We use Lemma 2 to do a straightforward case analysis of the candidate edges present in a planar drawing. If present candidate edges of A are all drawn on the same side of the cycle, then present candidate edges of B must be drawn on the other side, and there is no way to draw at least two candidate edges of C without crossing. The other possibility is that A has one candidate edge drawn inside and one outside the cycle. Then the same holds for B, and again, there is no way to draw the candidate edges of C without crossings, a contradiction. □

3.3 Proof of Theorem 3, part 2: Getting a c-planar drawing from G_1^M and G_2

In this section we show that if G_2 is without triangles and G_1^M is bipartite, then G has a planar saturator F; in other words, we can construct a supergraph G^F of G on the same set of vertices such that G^F is planar and for each cluster $A \in \mathcal{C}$ the graph $G[A]$ is connected. Actually, we directly construct an embedding of G^F into the plane. To this end, we first need the following Lemma:

Lemma 6 *Let G_1^M be bipartite and G_2 without triangles. Then for every connected component of G_2 with at least three vertices $y_{A_1}, \ldots, y_{A_k}$, one of the following cases occurs:*

- *there exist three groups g_1, g_2, $g_3 \in G_1^M$ such that for each cluster A_i, $i \in \{1, \ldots, k\}$, it holds that $g(A_i) \in \{g_1, g_2, g_3\}$, or*
- *all the vertices of all clusters $A_1, \ldots, A_k$ are within a single group in G_1^M.*

Proof: First, we show that the statement of the lemma holds for arbitrary three vertices y_A, y_B, y_C of G_2 that induce a connected subgraph in G_2. Suppose without loss of generality that $y_A y_B$ and $y_B y_C$ are edges in G_2. As G_2 has no triangles, we have that $y_A y_C$ is not an edge of G_2. By definition of G_2 it follows that A and B alternate and B and C alternate. Rule 5 causes that a_1, b_1, c_1 are merged into a group g_1, that a_2, b_2, c_2 are merged into a group g_2, and a_3, b_3, c_3 are merged into a group g_3. Lemma 4 asserts that A and C intersect, and hence by rule 3 there is an edge $g_1 g_3$ in G_1^M. If no further merging happens among g_1, g_2, g_3, and g_1, g_2, g_3 are three distinct groups, we are in the first case. If it happens that some two of the groups g_1, g_2, g_3 are merged, then it may occur that g_2 is merged with g_1 or g_3. Then all groups containing vertices of cluster A become non-isolated and by rule 6 we merge also g_1 and g_3. The other possibility is that g_1 and g_3 are merged, but then G_1^M contains a loop. A contradiction with the assumption that G_1^M is bipartite.

Application of the argument from the previous paragraph to the connected component of G_2 is now easy. Formally, we prove the statement by induction on the number of vertices. We actually prove a slightly stronger statement (which is useful to simplify the induction) that whenever $y_{A_1}, \ldots, y_{A_k}$ induce a connected subgraph in G_2, then the statement of the lemma holds for the corresponding

clusters. For components with three vertices, the statement was shown in the previous paragraph. So suppose we have $k > 3$ vertices $y_{A_1}, \ldots, y_{A_k}$ that induce a connected subgraph in G_2. Clearly, there is some $i \in \{1, \ldots, k\}$ such that the subgraph of G_2 induced on vertices $y_{A_1}, \ldots, y_{A_{i-1}}, y_{A_{i+1}}, \ldots, y_{A_k}$ is connected. Therefore, by induction, clusters $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_k$ satisfy the statement of the lemma and their vertices are either merged into three groups g_1, g_2, g_3 in G_1^M or into a single group G_1 of G_1^M. Vertex y_{A_i} is connected to some vertex y_{A_j}, which is also connected to some other vertex $y_{A'_j}$. Therefore the argument from the previous paragraph can be applied to $y_{A_i}, y_{A_j}, y_{A'_j}$, and we conclude that groups of cluster vertices of $A_i, A_j, A_{j'}$ are merged as described in the statement of the lemma. Therefore the groups of vertices of A_i are also merged in the groups g_1, g_2, g_3 or in the single group g_1, respectively. □

Lemma 7 *If G_2 is triangle-free and G_1^M is bipartite, then $(G, \mathcal{C})$ has a planar saturator F.*

Proof: Let I be the set of isolated vertices of G_1^M. Let us fix a drawing of the cycle G into the plane and some bipartition of $G_1^M \setminus I$ for the rest of this section. As G is a cycle, any drawing of G has well-defined inner and outer face, so drawing an edge of G^F inside or outside of the cycle is well defined. The idea behind our drawing is that edges represented by non-isolated vertices of G_1^M in the first part (we call it the *inner part*) are drawn inside the cycle and edges in the second part (called the *outer part*) are drawn outside the cycle. Vertices of I do not impose any restriction and therefore the edges represented by them can be drawn both inside or outside the cycle.

Now we present the construction of G^F and its drawing in more detail. We say a candidate edge a_1a_2 of G^F is *consistent with bipartition* if:

- Either $g_{A,a_1} \in I$ and $g_{A,a_2} \in I$, or
- exactly one of groups g_{A,a_1}, g_{A,a_2} is in I and the other group is in the inner part and the candidate edge is drawn inside the cycle, or
- exactly one of groups g_{A,a_1}, g_{A,a_2} is in I and the other group is in the outer part and the candidate edge is drawn outside the cycle, or
- g_{A,a_1} and g_{A,a_2} are in the inner part and the candidate edge is drawn inside the cycle, or
- g_{A,a_1} and g_{A,a_2} are in the outer part and the candidate edge is drawn outside the cycle.

Let $\mathcal{D}$ be a set of clusters represented by a connected component of G_2. We define graph G^F and the drawing of edges in these clusters for the whole component at once. We distinguish three cases:

Case 1. $|\mathcal{D}| = 1$: In this case $\mathcal{D}$ contains a single cluster $A = \{a_1, a_2, a_3\}$ that does not alternate with any other cluster. If $g_{A,a_1} = g_{A,a_2} = g_{A,a_3}$, we add to G^F candidate edges a_1a_2 and a_2a_3 and draw them consistently

with the bipartition. Otherwise at least one of the vertices g_{A,a_1}, g_{A,a_2} or g_{A,a_3} is isolated. Let it without loss of generality be g_{A,a_3}. In this case, we add to G^F candidate edges a_1a_3 and a_2a_3 and draw these edges consistently with bipartition.

Case 2. $|\mathcal{D}| = 2$: In this case $\mathcal{D}$ contains two clusters $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ that alternate but no other cluster alternates with any of them. We distinguish five cases:

2(a) There are a_i, b_j, with $i, j \in \{1, 2, 3\}$, such that g_{A,a_i} and g_{B,b_j} are non-isolated and belong to the same part. Note that a_i and b_j are not consecutive because of rule 4. Without loss of generality we set that $i = j = 1$, and the ordering of vertices in the cycle is a_1, b_3, a_2, b_1, a_3, and b_2. If both groups g_{B,b_3} and g_{A,a_2} are non-isolated, then by rule 4 there would be a path g_{A,a_1}, g_{B,b_3}, g_{A,a_2}, g_{B,b_1} in G_1^M and hence g_{A,a_1} cannot be in the same part as g_{B,b_1}, a contradiction. Hence one of g_{B,b_3} and g_{A,a_2} is isolated and by an analogous reason also one of g_{B,b_2} and g_{A,a_3} is isolated. If both g_{A,a_2} and g_{A,a_3} were non-isolated, then by rule 6 $g_{A,a_1} = g_{A,a_2} = g_{A,a_3}$ and the group is connected by an edge with g_{B,b_1} in G_1^M due to rule 4. Again a contradiction with the fact that g_{A,a_1} and g_{B,b_1} are in the same part. Similarly, we argue that one of g_{B,b_2} and g_{B,b_3} is isolated. Therefore we know that there is $i \in \{2, 3\}$ such that g_{A,a_i} and g_{B,b_i} are isolated. Without loss of generality we can assume that $i = 2$. In this case we add edges a_1a_2 and b_1b_2 to G^F and draw them consistently with bipartition. We also add edges a_2a_3 and b_2b_3 to G^F and draw them to the other face than a_1a_2 and b_1b_2 are drawn. Note that because of the rule 4 if g_{A,a_2} or g_{A,a_3} (or similarly g_{B,b_2}, g_{B,b_3}) are non-isolated, then they are in the other part than g_{A,a_1} (or g_{B,b_1}, respectively). Hence the resulting drawing is consistent with bipartition.

2(b) We are not in case 2(a) and there are $a_i, b_j, i, j \in \{1, 2, 3\}$ such that g_{A,a_i} and g_{B,b_j} are non-isolated and belong to the different parts. Without loss of generality we set that $i = j = 1$. Because we are not in case 2(a), all the groups g_{A,a_1}, g_{A,a_2}, g_{A,a_3} (and

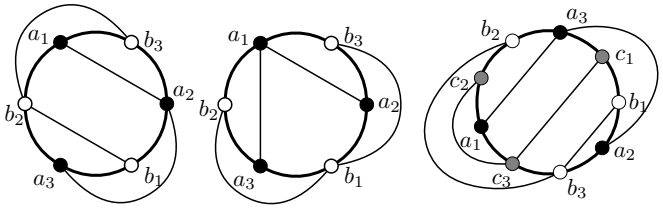


Figure 4: Drawings resulting from Case 2 and Case 3.

similarly g_{B,b_1}, g_{B,b_2}, g_{B,b_3}) are either isolated or belong to the same part. In this case we add a_1a_2, a_1a_3, b_1b_2 and b_1b_3 to G^F and draw them consistently with bipartition.

2(c) Two vertices among $g_{A,a_1}, g_{A,a_2}, g_{A,a_3}$ or among $g_{B,b_1}, g_{B,b_2}, g_{B,b_3}$ are non-isolated and all other vertices are isolated. Without loss of generality we assume that g_{A,a_1} and g_{A,a_2} are non-isolated and all other vertices g_{A,a_3}, g_{B,b_1}, g_{B,b_2}, and g_{B,b_3} are isolated. We can also assume that the ordering of vertices in the cycle is a_1, b_3, a_2, b_1, a_3, b_2. We add edges a_1a_3 and a_2a_3 to G^F and draw them consistently with bipartition. Observe that for each drawing of a_1a_3 and a_2a_3, there is exactly one possibility of drawing edges b_3b_1 and b_3b_2 so that they do not intersect a_1a_3 or a_2a_3. Thus we add these two edges to G^F and draw them this way.

2(d) There is exactly one non-isolated vertex among $g_{A,a_1}, g_{A,a_2}, g_{A,a_3}$, $g_{B,b_1}, g_{B,b_2}, g_{B,b_3}$. Without loss of generality let it be g_{A,a_1}. We add edges a_1a_2, a_1a_3 to G^F and draw them consistently with bipartition. We also add edges b_1b_2 and b_2b_3 to G^F and draw them to the other face than a_1a_2 and a_1a_3.

2(e) All of the vertices g_{A,a_1} , $g_{A,a_2}, g_{A,a_3}, g_{B,b_1}, g_{B,b_2}, g_{B,b_3}$ are isolated. In this case we add edges a_1a_2, a_1a_3 to G^F and draw them inside. We also add edges b_1b_2, b_1b_3 to G^F and draw them outside.

Case 3. $|\mathcal{D}| > 2$: In this case Lemma 6 asserts that either all the vertices of all clusters in $\mathcal{D}$ are in a single group g_1 or they are in three groups g_1, g_2, g_3. In the first case, for each cluster $A = \{a_1, a_2, a_3\} \in \mathcal{D}$ we add to G^F edges a_1a_2 and a_2a_3 and draw them consistently with bipartition. Note that in the second case there is exactly one edge among g_1, g_2, g_3 and one vertex is isolated (by Lemma 4 and due to rule 5). Without loss of generality g_1g_2 is the edge and g_3 is isolated. Hence we add edges a_1a_3 and a_2a_3 for all a_1, a_2, a_3 such that $g_{A,a_1} = g_1$, $g_{A,a_2} = g_2$ and $g_{A,a_3} = g_3$ to G^F and draw them consistently with bipartition.

Finally, if $A = \{a_1, a_2\}$ is a cluster of size two, we add a candidate edge a_1a_2 to G^F and draw it consistently with bipartition.

It is easy to check that after adding candidate edges to all clusters by the above rules, it will hold for all $A \in \mathcal{C}$ that $G^F[A]$ is connected. Hence we need only to check that the proposed drawing of G^F is planar. Note that all the edges are drawn consistently with bipartition. Suppose that the drawings of two candidate edges a_1a_2 and b_1b_2 from clusters A and B intersect. Clearly, clusters A and B must either intersect or alternate. If clusters intersect, then there are $i, j \in \{1, 2\}$ such that there is an edge $g_{A,a_i}g_{B,b_j}$ in G_1^M. Hence g_{A,a_i} and g_{B,b_j} belong to different parts and one of the edges a_1a_2 or b_1b_2 is drawn inconsistently with the bipartition, a contradiction. Suppose that clusters A and B alternate. If no other cluster alternates with A or B, then we applied case 2. It is easy to check that in all five subcases we drew edges so that they

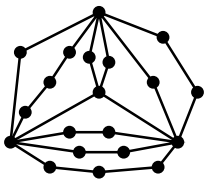


Figure 5: Example of a Rib-Eulerian graph created from K_4.

cannot intersect (see Figure 4). Finally, if we applied case 3, it is easy to verify that by the rule 6 no two edges can intersect (see Figure 4). □

Now we are ready to prove Theorem 3.

Proof: [of Theorem 3] By Lemma 2, a pair $(G, \mathcal{C})$ satisfying the assumptions is c-planar if and only if it has a planar saturator. Then, Lemma 5 and Lemma 7 provide the rest of the proof. □

Theorem 4 *Let G be a cycle and let $\mathcal{C}$ contain only clusters of size at most three. Then the c-planarity of $(G, \mathcal{C})$ can be decided in time $O(n + |\mathcal{C}|^3)$.*

Proof: The number of clusters is bounded by the number of vertices. We analyze individual steps of the algorithms.

1. The construction of G_2 according to "Definition of G_2" takes at most $O(|\mathcal{C}|^2)$.
2. The "Construction of G_1" takes at most $O(|\mathcal{C}|^2)$ (we proceed for each pair of clusters).
3. The "Construction of G_1^M" takes at most $O(|\mathcal{C}|^3)$.
4. We may merge at most linearly many times (we have linearly many vertices in G_1). During each merging we are manipulating only with edges incident to merged vertices, and each vertex has at most linearly many neighbors.
5. Bipartiteness can be checked in time linear to number of edges of checked graph (which is $O(|\mathcal{C}|^2)$).
6. Checking whether G_2 is triangle-free takes at most $O(|\mathcal{C}|^3)$ (we proceed for each triple of clusters).

□

4 Three-clusters on Rib-Eulerian graphs

The case of a single cycle, solved in Section 3, can be generalized to multiple cycles—faces of a plane graph. The "outside" of a face is the union of "insides"

of all neighboring faces. We will bicolor the faces so that the "outside" of a red face will be blue and vice versa. But, in order for this to work, we need the dual of the graph to be bipartite. Hence, the graph must be Eulerian.

Another possible obstacle to this approach are candidate edges that can be drawn into more than two faces. Then, there is no simple "inside" and "outside" for such an edge. Therefore, we limit the vertices of degree more than two to a constant number, and we treat them separately.

The suitable class of graphs is defined as follows. Let k be a constant; we call a graph *k-Rib-Eulerian* if it is Eulerian, and if it can be obtained from a 3-connected planar graph on k vertices by multiplying some edges, and then subdividing some edges. Figure 5 gives an example of such a graph.

We say that a path whose inner vertices have degree two and the outer vertices have degree larger than two is a *rib*. Thus a k-Rib-Eulerian graph consists of k vertices of degree at least four that are interconnected by ribs. A vertex of degree at least four is called a *branching vertex*. A cluster is called a *branch cluster* if it contains a branching vertex and either it has three vertices or it has two vertices which are both branching. Otherwise we call the cluster a *non-branch cluster*.

Let $(G, \mathcal{C})$ be a pair, where G is a Rib-Eulerian graph and $\mathcal{C}$ contains clusters of size at most three. We treat the branch clusters in a special way. Essentially, we try all the possibilities of choosing saturator edges for branch clusters. If the chosen edge connects two branching vertices, we add a pair (to preserve Eulericity) of edges connecting these two vertices to G. Otherwise we add a cluster of size two containing the two vertices to $\mathcal{C}$. As these clusters of size two can intersect, the final set $\mathcal{C}$ is not a cluster set but a pseudocluster set. For each choice of the saturator edges in the branch clusters we run the "Planar Saturator" algorithm on the resulting graph and pseudocluster set from which we remove all branch clusters. Clearly, the pair $(G, \mathcal{C})$ has a planar saturator if and only if it has a planar saturator for at least one of the saturator choices. Since a k-Rib-Eulerian graph has $O(k)$ branch clusters, there is a constant number of choices to check as long as k is constant.

In the "Planar Saturator" algorithm, we test the planarity of G. If G is planar, the planar embedding of the underlying 3-connected graph R on the sphere is unique. In order to find an embedding of G, we want to find the order of ribs of G originating from a common edge of R. This is done with respect to clusters in $\mathcal{C}$, because they force adjacencies of certain ribs. In this way, we obtain a sphere embedding of G.

If a suitable sphere embedding of G is obtained, we want to find a planar saturator. We utilize the algorithms of Section 3 that deal with cycles. In a sphere embedding of a Rib-Eulerian graph, the boundary of each face is a cycle. All the restrictions for candidate edges on a cycle apply in this case as well. And even more restrictions appear in this case, since "the outside" of faces is more complex. Basically, we construct the auxiliary graphs G_1 and G_1^M for the whole graph at once, applying the rules from the previous section on (parts of) clusters lying in the same face. The graph G_2 is constructed for each face separately. We then reduce the existence of a planar saturator of $(G, \mathcal{C})$ to the

bipartiteness of the graph G_1^M and the triangle-freeness of the graphs G_2 for each face.

Having found a planar saturator, the conditions of Lemma 8 determine whether a c-planar drawing of $(G,\mathcal{C})$ (in the plane) exists. We remark that the algorithm may be extended in a straightforward way to return not only the existential answer, but also a c-planar drawing. This, however, introduces further details and technicalities; for simplicity, we present the decision algorithm only.

Here we give an outline of the algorithm. Sections 4.1 and 4.2 contain more details of the particular steps, and Section 4.3 contains proofs of the correctness and the time complexity of the algorithm.

Definition 3 *Let $G=(V,E)$ be a k-Rib-Eulerian graph that was obtained from a 3-connected planar graph R and let $\mathcal{C}$ contain only clusters of size at most three. Let $E'=\{A\in\mathcal{C}:|A|=2\}$, $G'=(V,E\cup E')$. We say that $A\in\mathcal{C}$ is a* malicious triangle *if $|A|=3$, $G'[A]$ is a triangle and A contains only branching vertices.*

The augmentation of G with respect to $\mathcal{C}$ *is the graph* $\mathrm{aug}(G,\mathcal{C})$ *created from G by adding a vertex for each malicious triangle A and connecting this vertex by a pair of edges to each of the three vertices of A.*

Algorithm: C-planarity for Rib-Eulerian Graphs
Input: A Rib-Eulerian graph G; a cluster set $\mathcal{C}$ containing only clusters of size at most three
Output: `YES`, if the pair $(G,\mathcal{C})$ is c-planar; `NO` otherwise.

1. Construct the graph $H=\mathrm{aug}(G,\mathcal{C})$.
2. Find the underlying 3-connected graph R of H.
3. Test if there is a malicious triangle that is a 3-cut in R. If there is, return `NO`.
4. For all possible choices of saturator edges from branch clusters do:
   (a) $H':=H$; $R':=R$
   (b) Let $\mathcal{C}'$ be all non-branch clusters from $\mathcal{C}$.
   (c) For all chosen edges do:
      i. If the chosen edge connects two branching vertices, add a pair of edges connecting these two vertices to H'. Update R'.
      ii. Otherwise add a cluster of size two containing the two vertices to $\mathcal{C}'$.
   (d) Run the algorithm "Planar Saturator" on $(H',R',\mathcal{C}')$.
   (e) If the algorithm returned `YES`, then return `YES`.
5. Return `NO`.

Algorithm: Planar Saturator
Input: A Rib-Eulerian graph G; the underlying 3-connected planar graph R; a pseudocluster set $\mathcal{C}$ containing only non-branch clusters of size at most three
Output: `YES`, if the pair $(G,\mathcal{C})$ is has a planar saturator; `NO` otherwise.

1. If G is not planar, return `NO`.
2. Run the "Rib Ordering" algorithm on $(G,\ \mathcal{C},\ R)$ to obtain a sphere embedding of G (see Subsection 4.1 for details). If it fails, return `NO`.
3. Check if every cluster vertex shares a face with at least one vertex of the same cluster. If not, return `NO`.
4. Run the "Subclustering" algorithm on $(G,\ \mathcal{C})$ to obtain $\mathcal{C}'$, where $\mathcal{C}' \supseteq \mathcal{C}$. (details in Subsection 4.2)
5. Run the "Construction of G_1" algorithm on $(G,\mathcal{C}')$. (see Subsection 3.1 for description and 4.2 for comments on usage)
6. Run the "Forcing" algorithm on $(G,\ G_1,\ \mathcal{C}')$. (for details, see Subsection 4.2)
7. Run the "Construction of G_1^M" algorithm. (see Subsection 3.1 for description and 4.2 for comments on usage)
8. Run the "Construction of G_2" for each face separately to construct G_2^f. (see details in Subsection 3.1)
9. Check if G_1^M is bipartite and G_2^f are triangle-free for all faces f. If so, return `YES`, otherwise return `NO`.

4.1 Rib-ordering algorithms

This subsection aims to find a suitable sphere embedding of a planar Rib-Eulerian graph G with respect to the pseudocluster set $\mathcal{C}$. The reader may notice that in a possible c-planar embedding of $(G,\mathcal{C})$, the choice of the outer face is important. Hence a sphere embedding does not suffice in general. But for the purposes of finding a planar saturator, the choice of the outer face is irrelevant, so we work with a sphere embedding here.

First we concentrate on Theta graphs, i.e., graphs with two branching vertices only. For less than four ribs the embedding is unique; so let G have ribs $P_1, P_2, \ldots, P_s$ with $s \geq 4$.

We define a graph B_0 with vertices $\{1, 2, \ldots, s\}$, and edges $\{i, j\}$ for all $i \neq j$ such that there exists a cluster whose vertices are exactly on P_i and P_j. The edge $\{i, j\}$ expresses the fact that P_i and P_j are drawn next to each other.

Then we establish a 3-uniform hypergraph H on the same vertex set. The hyperedges of H are triples of ribs occupied by a common cluster. These rib

triples must also be drawn consecutively, in a yet unknown order. Formally, the hyperedges are $\{i, j, l\}$ such that there exists a cluster having one vertex on P_i, one vertex on P_j, and one vertex on P_l.

The algorithm "Rib Ordering for Theta Graphs" constructs a graph B from B_0 by adding new edges based on hyperedges of H. It looks for vertices that cannot be placed on the middle rib among three ribs determined by a particular hyperedge. Then it creates an ordering O of $V(B)$ (which defines a rib ordering of G straightforwardly).

Definition 4 *We say that h_1 is* a corner *of a hyperedge $\{h_1, h_2, h_3\} \in E(H)$ if any of following conditions is satisfied:*

- *h_1 has a neighbor in B different from h_2 and h_3.*
- *there is a hyperedge $\{h_1, h_4, h_5\} \in E(H)$ such that $\{h_4, h_5\} \cap \{h_2, h_3\} = \emptyset$.*
- *there is a cluster C in $\mathcal{C}$ having one vertex on each path P_{h_1}, P_{h_2} and P_{h_3} and another cluster $D \in \mathcal{C}$, such that on P_{h_1}, two vertices of D surround the vertex belonging to C.*
- *h_2 and h_3 are adjacent in B.*

The aim of this notion is to say, given a cluster with vertices spread among three different ribs, which rib cannot appear between the other two. If a vertex representing a particular rib becomes a corner of a hyperedge corresponding to a particular cluster $\mathcal{C}$, it means that this rib cannot be represented as the middle one among ribs relevant for representing $\mathcal{C}$. This intuition of property of corners is formally proved in the proof of Proposition 1. Note that the property of being a corner of a hyperedge depends on the graph B. For the sake of simplicity, in the following we consider the graph B to be implicit (although we should always say h is a corner of a particular hyperedge with respect to B). As we are constructing B only by adding edges, note that once a vertex becomes a corner of a particular hyperedge, it remains a corner of it, forever.

Proposition 1 *Let G be a Theta graph and let $\mathcal{C}$ contain only non-branch clusters of size at most three.*

1. *If the algorithm "Rib Ordering for Theta Graphs" finishes with an ordering O, then there is a clustered planar embedding of $(G, \mathcal{C})$ if and only if there is a clustered planar embedding of $(G, \mathcal{C})$ in which the ribs are ordered according to O.*
2. *If the algorithm "Rib Ordering for Theta Graphs" fails, then B contains either a non-Hamiltonian cycle or a vertex of degree greater than two. Then, $(G, \mathcal{C})$ is not c-planar.*

Proof: 1. We prove the "only if" implication because the other one is obvious.

Algorithm: Rib Ordering for Theta graphs

1. Start with the graph $B := B_0$ and the hypergraph H described above.
2. Apply the following rules as long as possible:

   Rule 8. If there are two hyperedges $\{h_1, h_2, h_3\}$, $\{h_2, h_3, h_4\} \in E(H), h_1 \neq h_4$ and $s > 4$ then add edge $\{h_2, h_3\}$ to B.

   Rule 9. If h_1 is a corner of a hyperedge $\{h_1, h_2, h_3\} \in E(H)$ then add edge $\{h_2, h_3\}$ to B.

   Rule 10. If none of the previous rules can be applied and there is a hyperedge $\{h_1, h_2, h_3\} \in E(H)$ such that $B[\{h_1, h_2, h_3\}]$ is not connected, then pick any such hyperedge h with the maximum number of corners, any two vertices $h_i \neq h_j$ of h that are non-adjacent in B, and add edge $\{h_i, h_j\}$ to B.

3. Check whether B is a union of vertex disjoint paths or a Hamiltonian cycle.

   If yes, create an arbitrary ordering O of $V(B)$ in which every two adjacent vertices are consecutive.

   If not, return `FAIL`.

In any clustered planar embedding *emb* of $(G, \mathcal{C})$, the embedding of each cluster C must be disjoint with all ribs except for those containing vertices of C. Hence, the ribs occupied by C must be drawn consecutively for every C.

We first show that there exists a clustered planar embedding of $(G, \mathcal{C})$ in which, for every edge $\{i, j\}$ in B, the ribs P_i and P_j are drawn next to each other. We proceed by induction on the construction of B. If $\{i, j\}$ is an edge of B_0, then there exists a cluster whose vertices are exactly on P_i and P_j, and hence the ribs P_i and P_j must be drawn next to each other in the c-planar embedding which we assume to exist.

Assume that there exists a clustered planar embedding *emb* in which the claim holds for all edges added to B before a certain step. There are several possibilities for the newly added edge $\{i, j\}$.

- An edge added by rule 8. If the rib P_{h_1} was drawn between P_{h_2} and P_{h_3}, then there would be no way to place P_{h_2}, P_{h_3} and P_{h_4} to make these three consecutive (because there are more than four ribs). So P_{h_2} and P_{h_3} are next to each other in any clustered planar embedding, hence also in *emb*.

- An edge added by rule 9. We show that if h_1 is a corner of a hyperedge $\{h_1, h_2, h_3\}$ then the rib P_{h_1} cannot be the drawn between the other two in *emb*. By Definition 4, if h_1 is a corner, then at least one of the following conditions holds.

- The vertex h_1 has a neighbor h_x in B different from h_2 and h_3. Then, by the induction assumption, P_{h_1} is always drawn next to P_{h_x}, and hence it cannot be between P_{h_2} and P_{h_3}.
- There is a hyperedge h_1, h_4, h_5 (with h_2, h_3, h_4 and h_5 mutually different). Since the ribs of each triple are consecutive in any clustered planar embedding, P_{h_1} must be next to at least one of P_{h_2} and P_{h_3}, and also next to at least one of P_{h_4} and P_{h_5}. Hence, it cannot be between P_{h_2} and P_{h_3}.
- There is a cluster C in $\mathcal{C}$ having one vertex on each path P_{h_1}, P_{h_2} and P_{h_3}, and a cluster $D \in \mathcal{C}$, such that on P_{h_1}, two vertices d and d' of D surround the vertex c belonging to C. The embedding of D is a connected region containing d and d'. The embedding of C is a connected region which contains c, one vertex of P_{h_2}, one of P_{h_3}, and it is disjoint with the embedding of D. If P_{h_1} is between P_{h_2} and P_{h_3}, this is not possible.
- If h_2h_3 is already an edge in B, then P_{h_2} and P_{h_3} are consecutive in *emb* by the induction hypothesis.

Hence P_{h_2} and P_{h_3} are also consecutive in *emb*.

- An edge added by rule 10. Let $h = \{h_1, h_2, h_3\}$ be the hyperedge used in rule 10. Since $B[h]$ is not connected, there is at most one edge in $B[h]$ and h has at most one corner.

  Assume that h_1 is a corner of h. Then $\{h_2, h_3\}$ is the only edge of $B[h]$. Without loss of generality, assume that we chose to add the edge $\{h_1, h_2\}$, while in the embedding *emb*, the ordering of the three ribs is $P_{h_1}, P_{h_3}, P_{h_2}$ (otherwise *emb* is the sought embedding).

  Suppose that there was a hyperedge h' containing exactly one of h_2, h_3 (say, h_2). If $h' \cap h = \{h_1, h_2\}$, then rule 8 can be applied to add $\{h_1, h_2\}$ to B, which contradicts the assumptions of rule 10. If $h' \cap h = \{h_2\}$, then h_2 is another corner of h, which is also a contradiction. Hence, any hyperedge different from h contains either both or none of h_2, h_3. Moreover, since h_2 and h_3 are not corners, they have no neighbors outside h (in B).

  It follows that if any cluster C has a vertex on one of P_{h_2} and P_{h_3}, then it has vertices on both of them. The embeddings of clusters must be disjoint, so the ordering of the cluster vertices along P_{h_2} is the same as along P_{h_3}.

  Hence we can obtain another correct clustered planar embedding from *emb* by switching the embeddings of P_{h_2} and P_{h_3}. Note that these switches occur for isolated edges $\{h_2, h_3\}$, so they are pairwise disjoint and nonconflicting.

  If h has no corner, then no other hyperedge may have nontrivial intersection with h. Hence, the vertices of every cluster occupy either all of the three ribs P_{h_1}, P_{h_3}, P_{h_2}, or none of them.

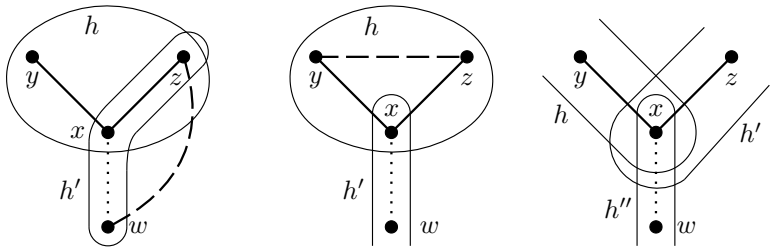


Figure 6: The possible cases of adding the third edge xw to a vertex x in B.

Assume again that we chose to add the edge $\{h_1, h_2\}$, while in the embedding *emb*, the ordering of the two ribs is $P_{h_1}, P_{h_3}, P_{h_2}$ or $P_{h_2}, P_{h_3}, P_{h_1}$.

Similarly as above, the embeddings of clusters must be disjoint in *emb*, so the ordering of cluster vertices along P_{h_2} is the same as along P_{h_3}. Hence we can switch the embedding of the ribs P_{h_2} and P_{h_3} and obtain a new clustered planar embedding from *emb* in which P_{h_1} and P_{h_2} are drawn next to each other.

Before this application of rule 10, the vertices h_2 and h_3 are isolated in B. During the application they become adjacent. Hence the switch of ribs P_{h_2} and P_{h_3} does not conflict with any other rib switches.

Let *emb* be the clustered planar embedding in which, for all edges of B, the corresponding ribs are adjacent. Assume that, in *emb*, there are non-consecutive ribs P_i and P_j, while i and j are consecutive in O. Then i and j are non-adjacent in B. If there is any non-edge in B, then B is a disjoint union of paths. No cluster contains vertices of more than one path, because rule 10 makes all hyperedge three-tuples connected.

Hence, the embeddings of rib groups corresponding to connected paths in B are disjoint (up to the two branching vertices). By switching and reverting the embeddings of rib groups, the sought clustered planar embedding can be obtained.

2. If the algorithm fails, then the graph B contains either a non-Hamilton cycle or a vertex of degree greater than two. For contradiction, let *emb* be a clustered planar embedding of $(G, \mathcal{C})$.

First assume that B contains a non-Hamilton cycle. We use the Jordan curve theorem. A non-Hamilton cycle in B implies a closed curve c (formed by clusters and parts of ribs) with one branching vertex inside and the other branching vertex y outside c. The non-Hamiltonicity of the cycle means that there are more ribs between the branching vertices, but no way to draw them without crossing c.

Now suppose that there is a vertex x of degree more than two in B. It follows from the proof of part 1 that for any edge $\{i, j\}$ of B_0, rule 8 or 9, the ribs P_i and P_j must be drawn next to each other in *emb*. But clearly no rib can

be next to more than two ribs.

Hence at most two edges incident with x originate from B_0, rule 8 or 9, and the remaining ones were added by rule 10. We discuss the situation when there are edges xy, xz, and the third edge wx is added by rule 10 (see Fig. 6).

Assume that there are hyperedges $h = \{x, y, z\}$ and $h' = \{x, w, u\}$. Assume also that $u = z$. The vertex x is a corner of h' because it is adjacent to y outside h'. But then, by rule 9, the vertices w and z are also adjacent, so $B[h']$ is already connected, which contradicts the condition of rule 10.

So assume that $u \neq y$, $u \neq z$. Then x is a corner of h because $h \cap h' = \{x\}$. So y and z are adjacent by rule 9, but then there is a triangle xyz, a contradiction.

The last case is that the three edges of x were added by distinct hyperedges whose intersection is only $\{x\}$. But then, in *emb*, the rib P_x is drawn next to one other rib of h, one other rib of h' and one other rib of h'', which is impossible. □

Let us consider a Rib-Eulerian graph G, a (pseudo)cluster set $\mathcal{C}$ and a subgraph H of G. We define *H-clusters* as those of $\mathcal{C}$ whose intersection with $V(H)$ is non-empty. We further define the *restriction of $\mathcal{C}$ to H* as the set $\mathcal{C}_H$ of H-clusters restricted to $V(H)$:

$$\mathcal{C}_H = \{C \cap V(H); C \in \mathcal{C}, C \cap V(H) \neq \varnothing\}.$$

Now consider the 3-connected graph R from which G was created. Each edge of R became a Theta graph in G. The planar embedding of R on the sphere is unique, hence the only choice for G is the ordering of ribs originating from each edge of R. That is done by the following algorithm "Rib Ordering" (see also Fig. 7 for illustration).

Proposition 2 *Let R be the 3-connected underlying graph of G and let $\mathcal{C}$ contain only non-branch clusters of size at most three. If there is a clustered planar embedding of $(G, \mathcal{C})$, then the "Rib Ordering" algorithm succeeds, and the resulting embedding can be extended to a c-planar embedding of $(G, \mathcal{C})$.*

Proof: Let *emb* be a clustered planar embedding of $(G, \mathcal{C})$. Clearly, the embedding φ of R is contained in *emb* (with a choice of the outer face).

Let xy be an arbitrary edge of R and let $(\vartheta, \mathcal{C}_\vartheta)$ be the instance created by the "Rib Ordering" algorithm for the edge xy. The graph ϑ consists of ribs P_0, $P_1, \ldots, P_{s+1}$; we denote the non-fake ribs $P_1, \ldots, P_s$ as *ϑ-ribs*.

Clearly, the c-planar embedding *emb* of the whole $(G, \mathcal{C})$ contains a correct embedding of all ϑ-ribs and clusters on them. As R is 3-connected, there are exactly two faces f_1 and f_2 in the embedding *emb* that are incident both to a ϑ-rib and a non-ϑ-rib. For both these faces, we now prove that their boundary in *emb* is "well represented" by the fake ribs P_0 and P_{s+1} of ϑ.

Let e be an edge and let r be the e-rib which was selected for ϑ as a part of a fake rib for the face f_1. Our aim is to prove that r is either adjacent to f_1 in *emb*, or there is a "very similar" rib adjacent to f_1 in *emb*.

Assume that the rib r was selected by criterion 1. Then there is a cluster $C \in \mathcal{C}$ having some vertices on a ϑ-rib, some vertices on r and no vertex on any

Algorithm: Rib Ordering
Input: A Rib-Eulerian graph G; a pseudocluster set $\mathcal{C}$; the 3-connected planar underlying graph R
Output: A sphere embedding of G, or `FAIL` if no suitable embedding is found

1. Find the unique sphere embedding φ of R.
2. For each edge $\{x, y\}$ in R do:
   (a) Let $P_1, P_2, \ldots, P_s$ be the ribs between x and y in G. Let $P = G[\bigcup_{i=1}^{s} P_i \setminus \{x, y\}]$ be the subgraph formed by them.
   (b) Let f_1 be a face of R incident with $\{x, y\}$. For each edge $e \neq \{x, y\}$ incident with f_1, select a rib r originating from e by the following criteria:
      Criterion 1. If there is a P-cluster having a vertex on a rib r, and having no vertex on any other rib of e, then select r.
      Criterion 2. If there are two ribs r_1 and r_2 of e such that all the P-clusters having vertex on some rib of e have vertices on both r_1 and r_2, then select r_1.
      Criterion 3. If the P-clusters have vertices on more than two ribs of e, or on none of them, then select an arbitrary rib of e.
      Concatenate the selected ribs (while preserving their order along f_1) into a new *fake*-rib P_0.
   (c) Let f_2 be the other face of R adjacent to $\{x, y\}$. In the same way, select ribs and concatenate them into a new fake-rib P_{s+1}.
   (d) Define a Theta graph ϑ consisting of ribs P_0, P_1, …, P_s, P_{s+1}, and let $\mathcal{C}_\vartheta$ be $\mathcal{C}$ restricted to ϑ. Call the algorithm "Rib Ordering for Theta Graphs" for the graph $(\vartheta, \mathcal{C}_\vartheta)$ with the edge $\{0, s+1\}$ added to B.
   (e) If the algorithm "Rib Ordering for Theta graphs" succeeds, let O be the returned ordering. Otherwise, return `FAIL`.
   (f) In the new embedding ψ, draw simple curves $l_1, \ldots, l_s$ instead of the edge $\{x, y\}$ in φ. Draw them close enough so that they do not interfere with the rest of the embedding ψ.
   (g) Assign the ribs $P_1, \ldots, P_s$ to the curves $l_1, \ldots, l_s$ according to the ordering O.
3. Return the embedding ψ.

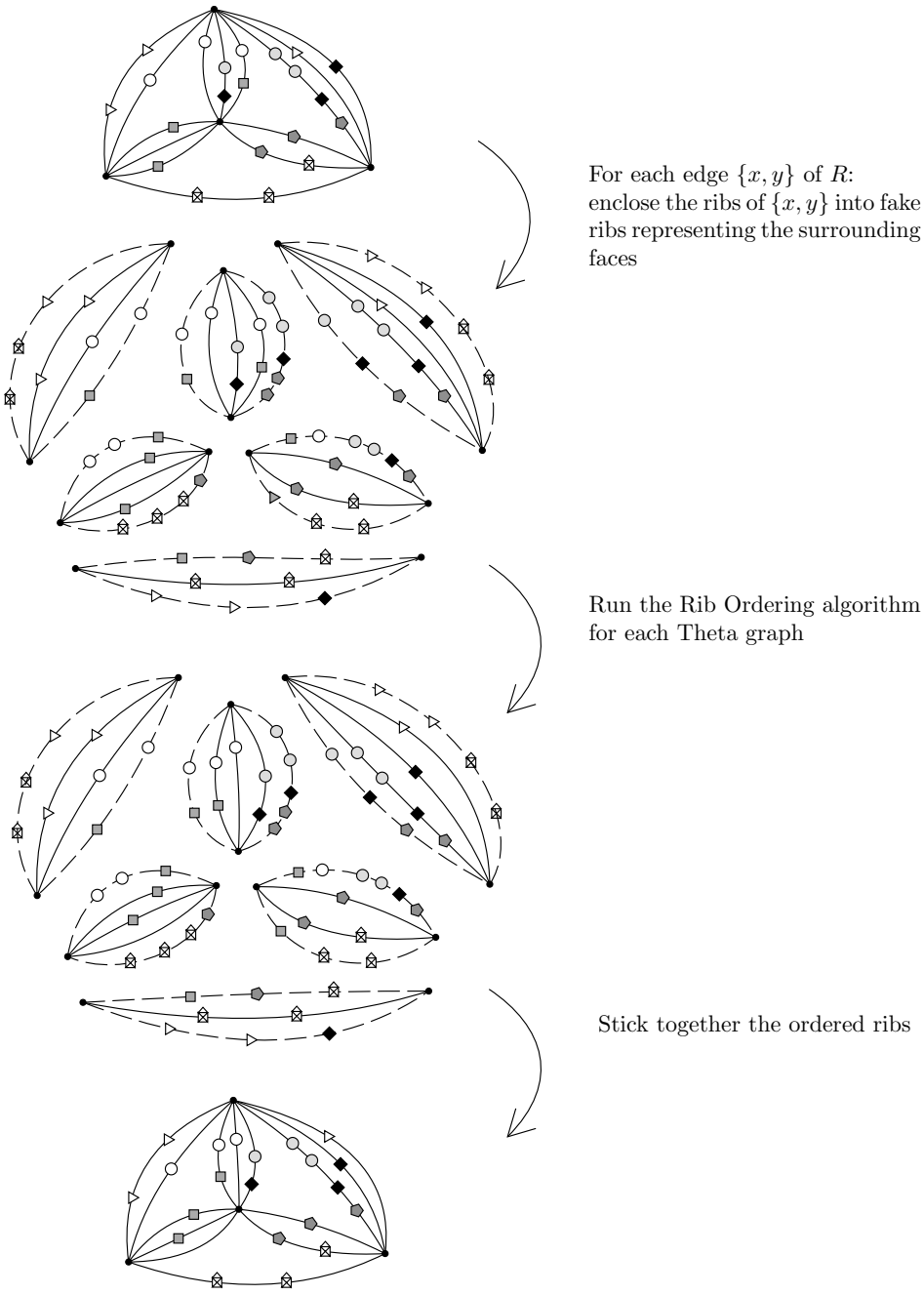


Figure 7: The "Rib Ordering" algorithm. Distinct symbols illustrate vertices of distinct clusters (except for branching vertices which do not belong to any cluster).

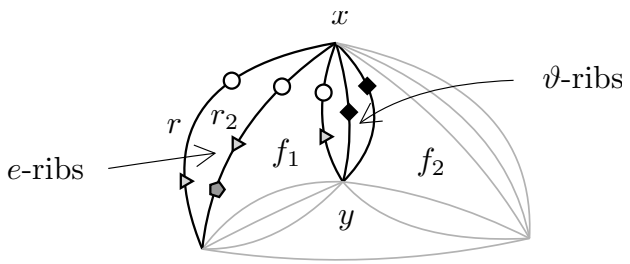


Figure 8: An illustration to the proof of Proposition 2.

other rib of e. As R is 3-connected, the degree of each vertex in R is at least three, and hence ribs of e and ϑ-ribs are adjacent to exactly one common face (say, f_1). Thus, it is clear that in *emb*, the rib r (containing vertices of C) is also adjacent to f_1, otherwise C would not be drawn correctly.

Assume that r was selected by criterion 2. Then there is a rib r_2 such that all ϑ-clusters have one vertex on r and one on r_2, and no vertex on any other rib of e (see Fig. 8). Then, by the above arguments, the rib adjacent to f_1 is either r or r_2. If the vertices of ϑ-clusters along r are in a different order than along r_2, then the areas of clusters in *emb* cannot be disjoint, which is a contradiction. Hence, r and r_2 contain vertices of ϑ-clusters in the same order. We call such a pair of ribs ϑ-*equivalent*.

Assume that r was selected by criterion 3. If the ribs of e contain vertices of more than two ϑ-clusters, then there are at least two ϑ-clusters whose vertices lie on different ribs of e. But only one rib of e is adjacent to f_1 in *emb*; hence, *emb* is not a c-planar embedding, which is a contradiction. Otherwise, ϑ-clusters do not have any vertex on any e-rib. Then r is ϑ-equivalent to the rib adjacent to f_1 in *emb*.

Hence, we may easily obtain a c-planar embedding of $(\vartheta, \mathcal{C}_\vartheta)$ from the embedding *emb* by restricting it to ϑ-ribs and boundaries of the faces f_1 and f_2. Thus, $(\vartheta, \mathcal{C}_\vartheta)$ is c-planar; by Proposition 1, the algorithm "Rib Ordering for Theta Graphs" succeeds for $(\vartheta, \mathcal{C}_\vartheta)$.

We have proved that, under the assumption that $(G, \mathcal{C})$ is c-planar, all calls of the algorithm "Rib Ordering for Theta Graphs" succeed. Then the "Rib Ordering" algorithm also succeeds and returns a sphere embedding ψ of G. It remains to prove that ψ can be extended to a c-planar embedding of $(G, \mathcal{C})$.

Since all calls of the algorithm "Rib Ordering for Theta Graphs" succeed, it follows from Proposition 1 that a c-planar embedding emb'_ϑ exists for every $(\vartheta, \mathcal{C}_\vartheta)$ instance.

The above arguments show that for each graph ϑ, its fake rib is composed of ribs that are ϑ-equivalent to those adjacent to f_1 and f_2 in the embedding *emb*. Now let P_i be the ϑ-rib adjacent to f_1 in the embedding emb'_ϑ (created by Theta Rib Ordering), and let P_j be the ϑ-rib adjacent to f_1 in *emb*. Since both *emb* and emb'_ϑ contain correct c-planar embeddings of all clusters in $\mathcal{C}_\vartheta$,

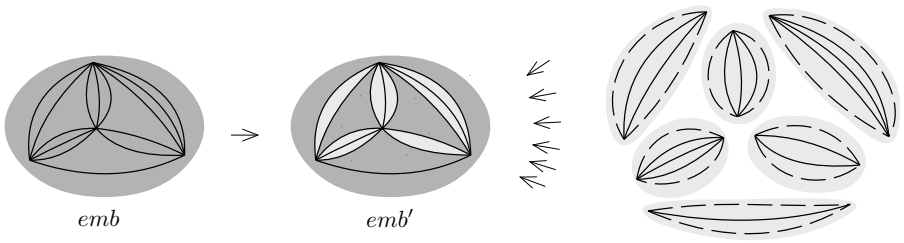


Figure 9: Combining the embeddings into emb', a c-planar extension of ψ.

the ribs P_i and P_j are also ϑ-equivalent. For the same reason, they are also θ-equivalent for other theta graphs θ adjacent to f_1.

Hence, the whole boundary of f_1 in emb'_ϑ is ϑ-equivalent to that in emb for every ϑ adjacent to f_1, and the same holds for f_2. This allows us to combine the embeddings emb'_ϑ with emb to obtain a new c-planar embedding emb' of $(G, \mathcal{C})$ in the way indicated in Fig. 9. Clearly, emb' is an extension of ψ returned by the "Rib Ordering" algorithm, which completes the proof. □

4.2 Construction of auxiliary graphs G_1, G_1^M and G_2

In the previous subsection we describe how to obtain a suitable sphere embedding of the input graph G. Now we fix such an embedding and seek a planar saturator of $(G, \mathcal{C})$.

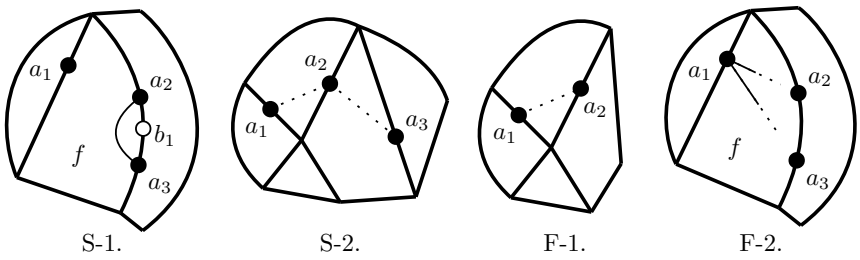
In the embedding, the boundary of each face is a cycle. The restrictions for candidate edges on a cycle apply here as well. Hence we might construct the auxiliary graph G_1 for each face as in Section 3 and then try to combine the results.

However, there are candidate edges which may be drawn in any of two adjacent faces. The choice of present candidate edges and the choice of their placement must be consistent in all faces. Therefore, we construct a single graph G_1 that represents the restrictions of all faces simultaneously.

Recall that in the cycle case, we use a vertex bicoloring of G_1 to determine the placement of present candidate edges. Here we do the same; but "the inside" does not have the same meaning for all faces. That is why we require the Eulericity of G.

We fix a proper bicoloring of faces of the fixed embedding of G by *red* and *blue*. Then, a candidate edge corresponding to a red vertex of G_1 is drawn in a red face. Since each candidate edge can be drawn in at most two adjacent faces (there are no branch clusters any more), and any two adjacent faces have distinct colors, the placement is well defined.

The constructions of G_1 and G_1^M are very similar to that in the cycle case. One difference is that the rules 1 through 7 deal only with tuples of clusters whose vertices all share a common face. Hence we introduce the "Subclustering"



S-1. S-2. F-1. F-2.

Figure 10: Illustration of rules of Subclustering (S-1 and S-2) and Forcing (F-1 and F-2) algorithms.

algorithm. It creates subclusters from candidate edges of clusters, whose vertices lie on the boundary of more than two faces.

Another difference in the construction of G_1 is that there may be more than a single face outside the considered face. Cluster edges whose endvertices lie on distinct ribs can be drawn in one face only. The "Forcing" algorithm represents this fact by forcing the corresponding vertices of G_1 to have a certain color in any possible bicoloring.

The graphs $G_2 = G_2^f$ are defined for each face f separately in the same way as in Section 3. This time, the *cyclic order* of vertices is taken with respect to the boundary of a face. The graphs G_1 and G_1^M are constructed for the whole graph at once, applying the rules on the clusters lying in each face.

Algorithm: Subclustering

Input: A Rib-Eulerian graph G; a pseudocluster set $\mathcal{C}$ containing only non-branch clusters of size at most three

Output: A pseudocluster set $\mathcal{C}' \supseteq \mathcal{C}$.

Initialization: $\mathcal{C}' = \mathcal{C}$

S-1. For every cluster $A = \{a_1, a_2, a_3\}$ whose two vertices a_2 and a_3 lie on a common rib different from that of a_1 (see Figure 10, part S-1), set $\mathcal{C}' = \mathcal{C}' \cup \{\{a_2, a_3\}\}$.

S-2. For every cluster $A = \{a_1, a_2, a_3\}$ lying in multiple faces so that the candidate edge $\{a_1, a_3\}$ cannot be drawn (see Figure 10, part S-2), set $\mathcal{C}' = \mathcal{C}' \cup \{\{a_1, a_2\}, \{a_2, a_3\}\}$.

Algorithm: Forcing
Input: A Rib-Eulerian graph G; the auxiliary graph G_1; a pseudocluster set $\mathcal{C}'$ containing only non-branch clusters of size at most three
Output: Modifies G_1.
Create two adjacent vertices v_{red} and v_{blue} of G_1 (the "forcing vertices").

F-1. For every 2-cluster A whose vertices lie on different ribs, and thus the candidate edge can be drawn in a unique face f only, create an edge between the vertex x_A of G_1 and the forcing vertex of the opposite color than the color of f (see Figure 10, part F-1).

F-2. For every cluster $A = \{a_1, a_2, a_3\}$ whose vertex a_1 lies on a different rib than a_2 and a_3, such that all the vertices lie in a face f, create an edge between the vertex x_{A,a_1} of G_1 and the forcing vertex of the opposite color than the color of f (see Figure 10, part F-2).

4.3 Main result

First we introduce a lemma that reduces the problem of c-planarity of a k-Rib-Eulerian graph to the problem of finding its planar saturator.

Lemma 8 *Let $G = (V, E)$ be a k-Rib-Eulerian graph that was obtained from a 3-connected planar graph R and let $\mathcal{C}$ contain only clusters of size at most three. Let $H = \text{aug}(G, \mathcal{C})$. Then H is an l-Rib-Eulerian graph, where $l = O(k)$. Furthermore, $(G, \mathcal{C})$ is c-planar if and only if:*

1. *There exists a planar saturator F of $(H, \mathcal{C})$ and*
2. *there does not exist a malicious triangle A that is a 3-cut in R.*

Proof: From the definition of augmentation it follows that H is created from G by adding one vertex and six edges per each malicious triangle. Since malicious triangles are clusters and hence cannot intersect, it is clear that the number of malicious triangles does not exceed k, the number of branching vertices. The vertices added to H are new branching vertices and the added edges are new ribs. Hence, H is an l-Rib-Eulerian graph, where $l = O(k)$.

Let $E' = \{A \in \mathcal{C} : |A| = 2\}$, $G' = (V, E \cup E')$ and let $\mathcal{C}'$ be the set of malicious triangles.

We prove that if $(G, \mathcal{C})$ is c-planar, then statements 1 and 2 hold. Since $(G, \mathcal{C})$ is c-planar, there exists a planar saturator F by Lemma 1 which is also a planar saturator of $(H, \mathcal{C})$ (it is certainly a saturator of $(H, \mathcal{C})$ and it is planar because we can draw each vertex for $A \in \mathcal{C}'$ inside the triangle formed by A). If there was a malicious triangle $A \in \mathcal{C}$ that is a 3-cut in R, then for each planar saturator F and every drawing of G^F, the triangle induced by A contains vertices not belonging to A. Thus every possible representation of cluster A will also contain vertices not belonging to A, and hence will not form a proper c-planar drawing. We conclude such cluster A cannot exist.

Now let us concentrate on the remaining implication—if statements 1 and 2 hold, then $(G, \mathcal{C})$ is c-planar. First we remove redundant edges from F. Let $A \in \mathcal{C}$ be such that $G^F[A]$ is a triangle. If there is an edge e in $G^F[A]$ that is not in $E \cup E'$, then $F' = F \setminus \{e\}$ is a planar saturator (by removing an edge we certainly keep planarity, $G^{F'}[A]$ still remains connected because $G^F[A]$ was a triangle, and if there is a 2-cluster $B \in \mathcal{C}$, $B \subseteq A$, B is also connected because $e \notin E'$). Thus we further assume that there are no such edges in F. Let $\mathcal{C}'' = \{A \in \mathcal{C} : G^F[A] \text{ is a triangle}\}$ (note that because F was reduced, these clusters are exactly the clusters where $G'[A]$ is a triangle). By Corollary 1 it is enough to create a representation of clusters in $\mathcal{C}''$. We start by constructing a sphere embedding ϕ of G^F from a sphere embedding of H^F (which is guaranteed to exist by the definition of F) by removing additional vertices of H. Each triangle $A \in \mathcal{C}''$ divides sphere into two connected areas. If both areas contain some vertices or edges, we call A *bad*. Because none of $A \in \mathcal{C}''$ is a 3-cut in R, one of the two areas of sphere can contain only some ribs parallel to one of the edges of A. Let us call this area f_A.

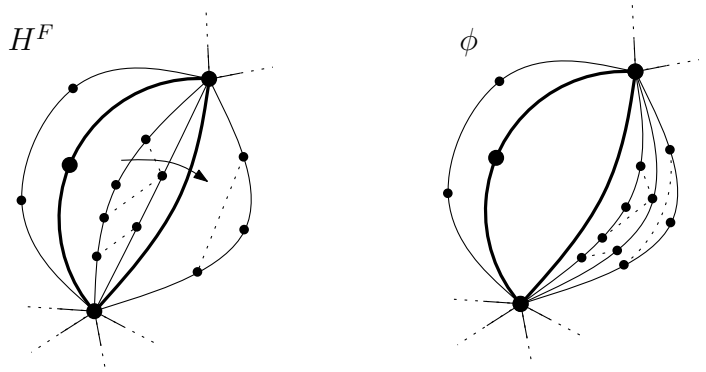


Figure 11: A bad triangle with one non-branching vertex in the embedding of H^F and in ϕ.

If A contains at least one non-branching vertex (see Fig. 11), there is at most one edge of A from which ribs inside f_A could be created and thus we can move all ribs out of the area creating new sphere embedding ϕ with less bad clusters.

Otherwise A contains only branching vertices (as in Fig. 12). In this case H contains an additional vertex connected to vertices of A and our sphere embedding of H^F embedded this vertex to an area f_A. Thus there are no edges between the ribs parallel to different edges of A in f_A. Hence we can move all the ribs out of f_A and obtain new sphere embedding ϕ with less bad clusters. We conclude that we can construct an embedding ϕ of G^F on the sphere containing no bad clusters. It only remains to pick proper outer face to obtain a planar embedding of G^F. We pick an arbitrary face that is not in $\mathcal{C}''$ as an outer face. In the obtained planar embedding of G^F, every cluster in $\mathcal{C}''$ forms a face and

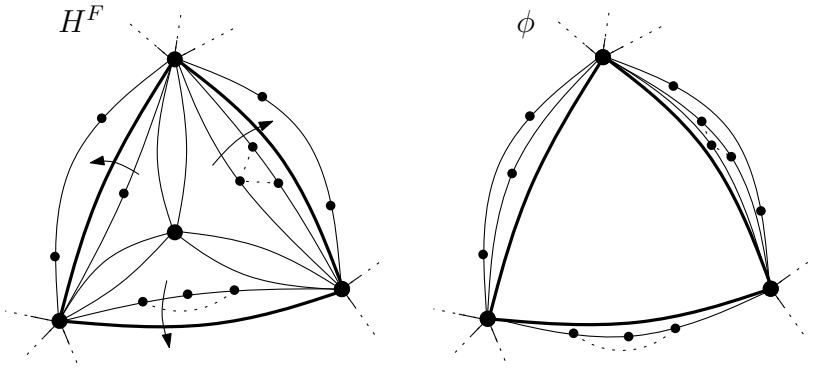


Figure 12: A malicious triangle in the embedding of H^F and in ϕ.

thus it is easy to create a representation for each cluster in $\mathcal{C}''$. □

For a pair $(G, \mathcal{C})$ and a face f of the fixed embedding of G, the symbol $(G, \mathcal{C}|_f)$ stands for the graph G and the set of clusters in $\mathcal{C}$ whose all vertices lie on the boundary of f.

We say that the graph G_1^M is *sparingly bicolorable* if it allows a proper bicoloring in which the group $g(v_{\text{red}})$ is red, the group $g(v_{\text{blue}})$ is blue, and the isolated vertices remain uncolored. We call such a coloring a *sparing bicoloring*.

The following proposition characterizes the pairs $(G, \mathcal{C})$ that have a planar saturator by means of the auxiliary graphs G_1^M and G_2^f.

Proposition 3 *The pair $(G, \mathcal{C})$ has a planar saturator if and only if G_1^M is sparingly bicolorable and the graph G_2^f is triangle-free for every face f.*

To prove the proposition, we need several lemmas. The following is an easy but important observation.

Observation 1 *For any fixed planar embedding of G, a candidate edge of a cluster in $\mathcal{C}'$ can be drawn without crossing in at most two adjacent faces.*

The observation holds because of the fact that $\mathcal{C}'$ contains only non-branching clusters. Lemma 9 below is an analogue of Lemma 3. It shows how the case of cycles may be generalized to faces of a k-Rib-Eulerian graph.

Lemma 9 *Let K be a planar saturator of $(G, \mathcal{C}')$. In a planar embedding of K into G, the following is true.*

1. *Let x and y be two adjacent vertices of G_1. Then any present candidate edge corresponding to x is drawn in a red face and any present candidate edge corresponding to y is drawn in a blue face, or vice versa.*
2. *For any non-isolated vertex v in $V(G_1^M) \setminus \{v_{\text{red}}, v_{\text{blue}}\}$, all present candidate edges corresponding to $g^{-1}(v)$ are drawn in faces of the same color.*

3. *Let x and y be two adjacent vertices in $V(G_1^M) \setminus \{v_{\text{red}}, v_{\text{blue}}\}$. Then any present candidate edge corresponding to $g^{-1}(x)$ is drawn in a red face and any present candidate edge corresponding to $g^{-1}(y)$ is drawn in a blue face, or vice versa.*

4. *Let z be a vertex in $V(G_1^M) \setminus \{v_{\text{red}}, v_{\text{blue}}\}$ adjacent to a forcing vertex v_{red} or v_{blue}. Then any present candidate edge corresponding to $g^{-1}(z)$ is drawn in a face whose color is blue or red, respectively.*

Proof: Note that the notation in the statement is correct, because the vertices v_{red} and v_{blue} do not belong to a cluster and thus are never merged with other vertices of G_1^M.

Let f be a face of the planar embedding of G. We say that a rule in the construction of G_1 and G_1^M was *applied in* f if the cyclic order of vertices in that rule was taken along this face.

We follow the construction of G_1 and prove inductively that part 1 holds after every step. Before any rule is applied, there are no edges in G_1 and it holds trivially. Then a step according to rule 1, 2, or 3 applied to a face f adds one new edge, say, xy. For all these rules, it is not hard to see that if an edge corresponding to x and an edge corresponding to y are drawn both inside f or both outside f, then they cross each other. Hence, one of them is inside f and the other one is in a neighboring face, which has the other color. Hence, after an application of rule 1, 2, or 3, part 1 remains valid.

Let us consider a step according to rule 4 applied to a face f. By Observation 1, each candidate edge can be drawn in at most two adjacent faces. Hence, any candidate edge corresponding to x or y is either drawn in f or in an adjacent face of the other color. Note that by the definition of rule 4, the vertices x and y are already non-isolated. Thus, by the validity of part 1 before this step, all present candidate edges corresponding to x are drawn in faces of the same color—either inside f, or in faces adjacent to f. The same holds for y. Then the same case analysis as in the proof of Lemma 3 finishes the proof of part 1.

Part 2 is proved by induction on the number of mergings taken by the construction of G_1^M. Before any merging was done, a non-isolated vertex v is either adjacent to another vertex of G_1, or to one of v_{red}, v_{blue}. In the former case, the claim holds as a consequence of part 1. In the latter case, it is easy to see that there is a unique face in which the candidate edges corresponding to v can be drawn without crossing. Hence, before any merging, the claim holds. We prove the validity of part 2 after each merging.

By the argument in the proof of Lemma 3, we may change the order of mergings without modifying the resulting graph G_1^M. Hence, we may assume that mergings of non-isolated groups were done before the first isolated group was merged. We analyze several cases.

(a) Merging due to rule 5. We further reorder the mergings so that all mergings of type (i) (see below) are done before the first one of type (ii).

   (i) Merging of g_{A,a_2}, g_{B,b_2} and g_{C,c_2}, or of g_{A,a_3}, g_{B,b_3} and g_{C,c_3}. If $g^{-1}(g_{A,a_2})$ is different from x_{A,a_2}, then g_{A,a_2} has already been merged

with something and it could be only through a merge of type (i) due to the order of merges. Hence g_{A,a_2} is non-isolated and by the induction hypothesis we know that all present cluster edges corresponding to g_{A,a_2} are drawn in faces of the same color. The same holds for g_{B,b_2} and g_{C,c_2}. By the same arguments as in the proof of Lemma 3 we prove that if a cluster edge corresponding to b_2 is drawn inside f, then all present candidate edges corresponding to a_2 and c_2 can only be drawn inside, too. Analogously, if a cluster edge corresponding to b_2 is drawn in a face adjacent to f, then all of a_2 and c_2 is also in faces adjacent to f. The desired conclusion follows.

(ii) Merging of g_{A,a_1}, g_{B,b_1} and g_{C,c_1}. If all g_{A,a_1}, g_{B,b_1} and g_{C,c_1} are isolated in G_1^M then there is nothing to prove. Assume without loss of generality that g_{A,a_1} is non-isolated. We know by the induction hypothesis that if both candidate edges corresponding to x_{A,a_1} are present, then they must be on the same side of f. Because g_{A,a_2} and g_{A,a_3} are now adjacent, we know again by the induction hypothesis that if any present candidate edge corresponding to x_{A,a_2} is inside f, then any present candidate edge corresponding to x_{A,a_3} is outside f and vice versa. Then a_2a_3 can not be present, and the edges a_1a_2 and a_1a_3 are on the opposite sides. But that is not possible by the restriction on x_{A,a_1}, and the instance is not c-planar, a contradiction.

(b) Merging due to rule 6. All three groups are non-isolated, and by the induction hypothesis and the same arguments as in the proof of Lemma 3, all present candidate edges corresponding to these groups are either inside f, or in faces adjacent to f which have the other color.

(c) Merging due to rule 7. By the induction hypothesis we know that all present candidate edges corresponding to the group g_{A,a_1} are drawn on one side of f, because g_{A,a_1} is non-isolated. If $g^{-1}(g_{A'})$ is different from $x_{A'}$ then it was merged in an execution of rule 7, so it is non-isolated. This means that the group $g_{A'}$ is drawn on one side of f, too. Analogously as in the proof of Lemma 3, we conclude that present cluster edges corresponding to g_{A,a_1} and $g_{A'}$ are drawn in faces of the same color.

Part 3 is an easy consequence of the first two and the fact that, if there is an edge between two vertices x and y of G_1^M, then there exists an edge between two vertices u and v of G_1, where $u \in g^{-1}(x)$ and $v \in g^{-1}(y)$.

To prove part 4, let z be a vertex of G_1^M adjacent to v_{red} (without loss of generality). The group $g^{-1}(z)$ contains a vertex adjacent to v_{red} in G_1 after the "Forcing" algorithm. Let the vertex be x. It is easy to see that there is a unique face in which the candidate edges corresponding to x can be drawn without crossing, and the face is blue. As x is non-isolated, the group z in G_1^M is also non-isolated. Hence by part 2, all present candidate edges corresponding to z are drawn in a blue face. □

Lemma 10 *If $(G, \mathcal{C})$ has a planar saturator, then G_1^M is sparingly bicolorable and the graphs G_2^f are triangle-free for all faces f.*

Proof: We observe that if the input pair $(G, \mathcal{C})$ has a planar saturator, then the pair $(G, \mathcal{C}')$ obtained by running the "Subclustering" algorithm has a planar saturator as well. Indeed, let e be a candidate edge of a two-cluster C added by the "Subclustering" algorithm to $\mathcal{C}'$. In case of rule S-2 it is clear that e is present in any planar saturator. In case of rule S-1 assume that $e = \{a_2, a_3\}$ is not present; then the other two candidate edges $\{a_1, a_2\}$ and $\{a_1, a_3\}$ of C are drawn in a common face (see Figure 10) and none of them is crossed by another edge. Moreover, there is no other cluster C_2 such that $C \cap C_2 = \{a_1\}$. Hence there is no other curve outgoing of the vertex a_1, and the edge e can be drawn by tracing the curves of $\{a_1, a_2\}$ and $\{a_1, a_3\}$. Therefore, we may further consider a planar saturator K of $(G, \mathcal{C}')$.

Let f be a face of G. Consider K restricted to f, i.e., the set of edges of K that have both endpoints on the boundary of f. Clearly, this is a planar saturator of $(G, \mathcal{C}'|_f)$. Hence, by Lemma 5, the graph G_2^f is triangle-free, which proves one part of the Lemma.

To prove the other part, suppose for contradiction that G_1^M is not bipartite. Let $(v_1, \ldots, v_{2k+1})$ be an odd cycle in G_1^M.

First let the cycle contain no forcing vertex. By Lemma 9 the present candidate edges corresponding to $g^{-1}(v_1)$ are drawn all in the same face f_1 of G; without loss of generality let it be red. Then again by Lemma 9 the present candidate edges corresponding to $g^{-1}(v_2)$ are drawn in a blue face f_2 adjacent to f_1, those of $g^{-1}(v_3)$ are drawn in a red face f_3 adjacent to f_2, etc. But then the present candidate edges corresponding to $g^{-1}(v_{2k+1})$ are drawn in a red face adjacent to the red face f_1, a contradiction with the proper face bicoloring.

Now let the odd cycle contain forcing vertices; without loss of generality let $v_1 = v_{\text{red}}$ and $v_2 \neq v_{\text{blue}}$. Then by Lemma 9 the edges corresponding to $g^{-1}(v_2)$ are all drawn in a blue face. Analogously, it can be done for v_{blue} and its neighbors, if v_{blue} is contained in the cycle. By repeated application of Lemma 9 for all pairs of adjacent vertices along the cycle we get a contradiction similar to the one above. □

Lemma 11 *Let f be a face of the fixed embedding of G. If G_1^M is sparingly bicolorable and G_2^f triangle-free, then $(G, \mathcal{C}'|_f)$ has a planar saturator.*

Proof: Let F be the boundary of the face f.

If G_1^M is sparingly bicolorable, then it is bipartite. We redo the steps of the proof of Lemma 7 using the auxiliary graphs G_1^M and G_2^f, and draw all clusters of $\mathcal{C}'|_f$ according to the sparing bicoloring. The inner part of f will have the same color as f; the outer part will have the other one. Since the faces are properly bicolored, this is consistent with the face colors—all faces adjacent to f have the opposite color to the one of f.

Lemma 7 ensures that we get a planar embedding of $(F, \mathcal{C}'|_f)$ with a saturator. In the following, we show that we obtain even a correct planar embedding of $(G, \mathcal{C}'|_f)$ with a saturator.

Suppose that the embedding of saturator edges in $(F, \mathcal{C}'|_f)$ is not correct in $(G, \mathcal{C}'|_f)$. In that case, a saturator edge e crosses an edge e' of G; without loss of generality let v be one of the endvertices of the edge e', and let v lie on the boundary of f.

The vertex v separates f into ribs. Each face contains more than one rib (because of the properties of G). Then the endvertices of e lie in different ribs of f, which is either case F-1 or F-2 (see Figure 10). Hence the "Forcing" algorithm has caused the edge to be drawn in the inner part by connecting a related vertex to a certain forcing vertex.

But the edge e' does not belong to face f, it is drawn outside f, hence the crossing of e and e' occurs outside of f. A contradiction. □

Proof: [of Proposition 3] The first implication is the statement of Lemma 10.

It remains to prove that if G_1^M is sparingly bicolorable and the graphs G_2^f are triangle-free for all faces f, then $(G, \mathcal{C})$ has a planar saturator. In the following, we find a saturator of $(G, \mathcal{C}')$ together with its embedding. Clearly, every saturator of $(G, \mathcal{C}')$ is a saturator of $(G, \mathcal{C})$ as well.

By Lemma 11, for each face f the pair $(G, \mathcal{C}'|_f)$ has a saturator K^f; the proof of the Lemma even finds an embedding of each G^{K^f}. We claim that by simply taking the union of all saturators K^f, we obtain a saturator K of the whole $(G, \mathcal{C}')$. We verify that for all clusters A, the graph $G^K[A]$ is connected. There are two possibilities for the vertices of A:

- All the vertices of A lie in a common face f. Then the cluster A is an element of $\mathcal{C}'|_f$ and Lemma 11 guarantees that the graph $G^{K^f}[A]$ is connected.

- The vertices of A do not lie in a common face. Then, as the algorithm "Planar Saturator" has not stopped unsuccessfully in step 3, the cluster A has three vertices contained in two adjacent faces. One of them, say, a_2, shares a face f with another vertex a_1, and it shares another face f' with a_3. In that case the pairs $\{a_1, a_2\}$ and $\{a_2, a_3\}$ were made 2-clusters by rule S-2, and those are connected in G^{K^f} and $G^{K^{f'}}$.

In both cases, we conclude that $G^{K^f}[A]$ is indeed connected. It remains to prove that the saturator K allows a planar embedding of G^K. Such an embedding is obtained by uniting all the embeddings of G^{K^f} and by deleting possible multiple drawings of edges.

We already know that the edges of any K^f do not cross the edges of G, hence the edges of K do not cross them either. It remains to verify that the edges of K do not cross each other.

Suppose that on the contrary, there are two edges $e \in K^f$ and $e' \in K^{f'}$ that cross each other. Then clearly $f \neq f'$. Because of the properties of G, no face is adjacent both to f and f'; hence the crossing of e and e' must occur in f or f'. Without loss of generality, let it be f.

We distinguish the following cases.

- Both e and e' are candidate edges of 2-clusters A and B. Then the cluster vertices of both clusters lie on the boundary of f, and their cyclic order along the boundary is *abab*. Then there is an edge between $g(A)$ and $g(A')$ in G_1^M, which is a contradiction with the correctness of the embedding of G^{K^f}.

- The edge e is a candidate edge of a 2-cluster A and e' is a candidate edge of a 3-cluster B. If there exists a 2-cluster C equal to e', then it is drawn identically to e' and we repeat the previous argument for e and the candidate edge of C.

  Otherwise, all the vertices of B lie on the boundary of a common face (because of rule S-2). If the face was f, then there would be an edge between $g(A)$ and $g(b_i)$ for a certain i, which would contradict the correctness of the embedding of G^{K^f}. However, all the vertices of B lie on a common rib of f', otherwise e' would be a 2-cluster (rule S-1); hence all the vertices of B lie on the boundary of f, a contradiction.

- Both e and e' belong to 3-clusters and there are no 2-clusters identical to any of them. Then all the vertices of each of them lie on the boundary of a common face (because of rule S-2), and it is one of the situations depicted in Figure 13.

  In each of those situations, rule S-2 created a 2-cluster equal to one of the crossing edges, which is a contradiction.

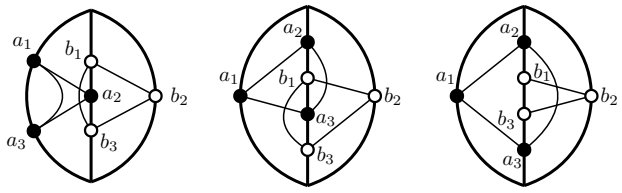


Figure 13: The possible cases of a multiple-face cluster conflict.

□

Theorem 5 *The c-planarity of $(G, \mathcal{C})$ can be decided in time $O(3^k \cdot n^3)$ for G being k-Rib-Eulerian with n vertices and $\mathcal{C}$ containing clusters of size at most 3.*

Proof: First we check whether the second condition from Lemma 8 is satisfied. This can be clearly done in time $O(|\mathcal{C}| \cdot k + n^2)$. If the condition is not satisfied, we know $(G, \mathcal{C})$ is not c-planar and reject. Otherwise Lemma 8 asserts that $(G, \mathcal{C})$ is c-planar if and only if $H = \text{aug}(G, \mathcal{C})$ has a planar saturator F. The graph H can be constructed in time $O(n)$.

When constructing planar saturator, we first deal with branch clusters. We try all possibilities of choosing saturator edges for branch clusters. If the chosen edge connects two branching vertices, we add a pair (to preserve Eulericity) of edges connecting these two vertices to H. Otherwise we add a cluster of size two containing the two vertices to $\mathcal{C}$. As these clusters of size two can intersect, the final set $\mathcal{C}$ is not a cluster set but a pseudocluster set. For each choice of the saturator edges in the branch clusters we run the "Planar Saturator" algorithm on the resulting graph and pseudocluster set from which we remove all branch clusters. Clearly, if $(H, \mathcal{C})$ has a planar saturator, then at least one of the choices of edges for branch clusters is its subset. Since there are at most k branch clusters in H and each branch cluster has at most 3 possible saturators, the algorithm "Planar Saturator" is called at most $O(3^k)$ times.

Before we count the time needed to execute the algorithm "Planar Saturator" itself, notice that $|C| = O(n)$. The planarity testing of the input graph with n vertices takes $O(n)$ time (see [11]). The algorithm "Rib Ordering for Theta Graphs" can be done in $O(n^2)$ time. H has at most n hyperedges and B has at most $O(n)$ edges due to Proposition 1(2). As each of rules 8, 9 and 10 adds some edge, there can be at most $O(n)$ applications of the rules. All the occurrences of rule 8 can be applied at once in $O(n^2)$ time. Then we decide which vertices are corners, and this takes $O(n^2)$ time. Afterwards, we apply rules 9 and 10; it always takes $O(n)$ time to find the occurrence, and then $O(n)$ to recount the corners. The consistence test can obviously be done in linear time.

The "Rib Ordering" algorithm for the entire graph can thus be done in $O(kn^2)$ time. There are at most n paths, and to construct the fake-paths we need $O(nk)$ time as we try each cluster. The call of the algorithm "Rib Ordering for Theta Graphs" always takes $O(n^2)$ time and we call it at most $O(k)$ times, because that is the number of edges of the original planar 3-connected graph. Proposition 2 verifies correctness of this step.

The test for the neighboring cluster vertex in the same face together with the "Subclustering" and "Forcing" algorithms can be done in $O(kn)$ for each cluster with the suitable representation of the embedding, thus it takes $O(kn^2)$ together.

As we proved in Theorem 4, the graphs G_1, G_1^M and G^2 can be constructed and tested for the bipartiteness and triangle-freeness in $O(n^3)$ time. Proposition 3 together with the above arguments give us that the algorithm "C-planarity for Rib-Eulerian Graphs" succeeds if and only if $(H, \mathcal{C})$ has a planar saturator which (according to Lemma 8) is if and only if $(G, \mathcal{C})$ is c-planar.

Altogether this gives an $O(3^k \cdot n^3)$ time complexity for the entire algorithm. □

5 Conclusion

We have proposed to study the special case of c-planarity when clusters contain a small number of vertices. We have presented polynomial-time algorithms for clusters of size at most three in vertex-3-connected planar graphs, on a cycle and

in Rib-Eulerian graphs. Thus, we have introduced a new class of polynomial c-planarity instances, which was not considered before. We believe that this may help to get more insight into the assumed boundary between polynomial and hard instances of c-planarity.

The algorithm for 3-clusters on a cycle is unexpectedly non-trivial. We have presented a generalization to the class of Rib-Eulerian graphs. However, there seems to be no straightforward generalization of this algorithm to a larger graph class; our method relies on the fact that the dual of the underlying graph is bipartite.

A possible generalization of our approach may be to larger clusters. Then, the auxiliary graphs of Section 3 would have to be extended to contain a vertex for every edge-cut of the candidate edges of each cluster. But then the number of cases to analyze grows very large. Hence, we suggest that more insight should be gained instead of generalizing this method straightforwardly.

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