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On the Biplanarity of Blowups

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Abstract. The 2-blowup of a graph is obtained by replacing each vertex with two non-adjacent copies; a graph is biplanar if it is the union of two planar graphs. We disprove a conjecture of Gethner that 2-blowups of planar graphs are biplanar: iterated Kleetopes are counterexamples. Additionally, we construct biplanar drawings of 2-blowups of planar graphs whose duals have two-path induced path partitions, and drawings with split thickness two of 2-blowups of 3-chromatic planar graphs, and of graphs that can be decomposed into a Hamiltonian path and a dual Hamiltonian path.

1 Introduction

In a 2018 survey on the Earth–Moon problem, Ellen Gethner conjectured that 2-blowups of planar graphs are always biplanar [14]. In this paper we refute this conjecture by showing that 2-blowups of iterated Kleetopes are non-biplanar, and more strongly do not have split thickness two.

1.1 Definitions and preliminaries

Before detailing our results, let us unpack this terminology: what are Kleetopes, biplanarity and split thickness, and blowups?

- *Polyhedral graphs* are the graphs of convex polyhedra. By Steinitz's theorem, these are exactly the 3-vertex-connected planar graphs [27]. Polyhedral graphs have planar embeddings that are unique up to the choice of outer face [23]. The faces of these embeddings are exactly the *peripheral cycles*, cycles such that every two edges not in the cycle are part of a path with interior vertices disjoint from the cycle [29]. Every *maximal planar graph* with ≥ 4 vertices, one to which no edges can be added while preserving planarity, is 3-vertex-connected [17], and therefore polyhedral, with triangular faces.

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The *Kleetope* of a polyhedral graph (named by Branko Grünbaum for Victor Klee [16]) is a maximal planar graph obtained by adding a new vertex within every face, adjacent to all the vertices of the face. Geometrically, it can be formed by attaching a pyramid to every face, simultaneously. An *iterated Kleetope* is the result of repeatedly applying this operation a given number of times. In previous work, we used iterated Kleetopes to construct polyhedral graphs that cannot be realized geometrically with isosceles triangle faces [11]. Following the notation from that work, let KG denote the Kleetope of a graph G and K^iG denote the result of applying the Kleetope operation i times to G.

- *Thickness* is the minimum number of planar subgraphs needed to cover all edges of a given graph. Equivalently, it is the minimum number of edge colors needed to draw the graph in the plane with colored edges so each crossing has edges of two different colors. A graph is *biplanar* if its thickness is at most two. Thus, a biplanar drawing of a graph can be interpreted as a pair of planar drawings of two subgraphs of the given graph that, together, include all of the graph edges. Repeated edges are never necessary and for technical reasons we forbid them.

  Little was known about general methods for showing that graphs are not biplanar, or for finding natural classes of graphs that are sparse enough to be biplanar but are not biplanar. One such class was provided by Sýkora et al., who observed that 5-regular graphs of girth at least 10 are too dense for their girth to be biplanar [28]. An NP-completeness reduction for biplanarity by Mansfield [24] can also be used to construct infinitely many non-biplanar graphs for which simple tests such as checking the sparsity of the graphs fails to determine that they are non-biplanar. All graphs of maximum degree four are biplanar [8, 18]; this limits the use of counting arguments to construct biplanar graphs, because (in contrast to planar graphs) the number of non-isomorphic degree-four graphs is not singly exponential in the number of vertices.

- *Split thickness* is a generalization of thickness in which we form a single planar drawing with multiple copies of each vertex, which are not required to be near each other in the drawing. Each edge of the graph appears once, connecting an arbitrary pair of copies of its endpoints. A drawing has split thickness k if each vertex has at most k copies, and the split thickness of a graph G is the minimum number k such that G has a drawing with split thickness k [12]. Split thickness is less than or equal to thickness, but they can diverge, even for complete graphs: K_{12} has split thickness two [19] but K_9 already has thickness three [2, 30]. Thickness has its origin in the Earth–Moon problem, posed by Gerhard Ringel in 1959 [26], which in graph-theoretic terms asks for the maximum chromatic number of biplanar graphs. In the same way, split thickness corresponds to the older m-pire coloring problem [13].

- *Blowups* of graphs are formed by duplicating their vertices a given number of times. More specifically, the (open) k-blowup of a graph G, which we denote as kG,[1] is obtained by making k copies of each vertex of G, and by connecting two vertices in kG whenever they are copies of adjacent vertices in G. Two copies of the same vertex are not adjacent. In a *closed blowup*, the copies are adjacent.

  Albertson, Boutin, and Gethner [1] proved that the closed 2-blowup of any tree or forest is planar and therefore that the closed 2-blowup of any graph of arboricity a has thickness at most a. Here, *arboricity* is the minimum number of forests that cover all edges of a graph.

[1] Blowups are a standard concept but their notation varies significantly. Other choices from the literature include $G(k)$, $G[k]$, G^k, and $G^{(k)}$.

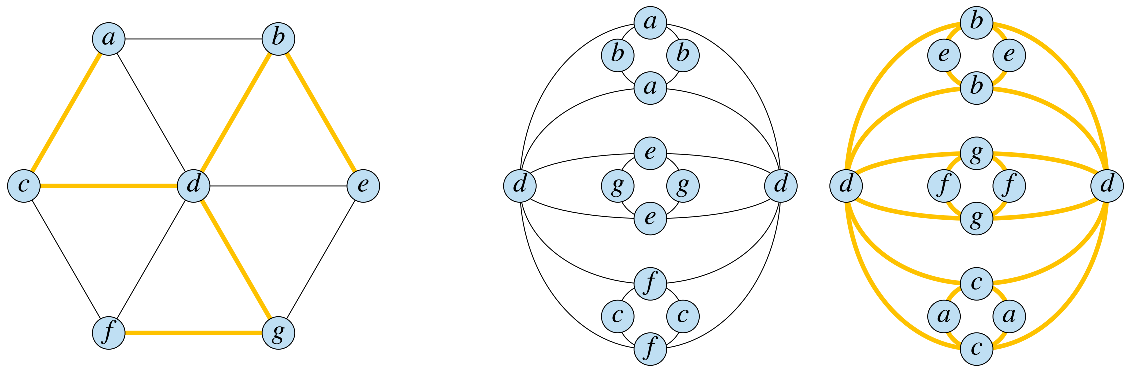

Figure 1: Decomposition of a seven-vertex wheel graph into two trees (left) and the corresponding biplanar drawing of its 2-blowup (right)

> Adding a leaf to a forest corresponds, in the closed 2-blowup, to gluing a K_4 subgraph onto an edge, which preserves planarity, and the a forests that cover a graph of arboricity a can be blown up by induction in this way, separately from each other (Fig. 1).

Planar graphs have arboricity at most three [25], and this is tight for planar graphs with more than $2n-2$ edges, including most maximal planar graphs. Therefore, their 2-blowups have thickness at most three. Gethner's conjecture asks whether smaller thickness, two, can always be achieved. We prove that it cannot: there exist planar graphs for which the thickness-three drawing of the 2-blowup, obtained using arboricity as above, is optimal.

1.2 New results

Our main result is that for all sufficiently large maximal planar graphs G, $2K^3G$ does not have thickness two and does not have split thickness two. This gives a proof of non-biplanarity for a natural and sparse class of graphs that (unlike previous methods for proving non-biplanarity) allows short cycles.

To complement this result, we provide biplanar or split thickness two drawings for the blowups of three natural classes of planar graphs:

- We construct a biplanar drawing of the 2-blowup of any planar graph whose faces can be decomposed into two *outerpaths*, strips of polygons connected edge-to-edge with the topology of a path (see Section 3). Equivalently, this structure is a partition of the dual graph into two induced paths.
- When G can be decomposed into two outerpaths, we construct a split thickness two drawing of its Kleetope KG. In the special case of the tetrahedral graph K_4, this construction can be used for the iterated Kleetope K^2K_4.
- When G and its dual have disjoint Hamiltonian paths, we construct a split thickness two drawing of the 2-blowup of G.
- We construct a split thickness two drawing of the 2-blowup of any 3-chromatic planar graph, and more generally a drawing with split thickness k of the k-blowup of these graphs.

These drawing algorithms motivate our use of iterated Kleetopes in constructing non-biplanar blowups of planar graphs, because Kleetopes are far from having the properties needed to make these algorithms work. Kleetopes of maximal planar graphs are far from 3-chromatic: each added vertex forms a K_4 subgraph, an obstacle to 3-coloring. And iterated Kleetopes are far from being decomposable into outerpaths, as their dual graphs have no long induced paths. The underlying planar graphs for each of our drawing algorithms include infinitely many maximal planar graphs, showing that it is not merely the large size and maximality of iterated Kleetopes that prevents their blowups from having drawings. Additionally, because triangle-free planar graphs are 3-chromatic [15], these constructions suggest that the connection of Sýkora et al. between girth and non-biplanarity is unlikely to help construct planar graphs with non-biplanar blowups.

In a final section of this paper we briefly address computational complexity issues involving testing the biplanarity of blowups.

2 Iterated Kleetopes

In this section we show that some 2-blowups of iterated Kleetopes are not biplanar and do not have split thickness two or less. As in our previous work on the geometric realization of iterated Kleetopes [11], our approach uses the observation that any realization or drawing of an iterated Kleetope must be based on a realization or drawing of a graph with one fewer iteration. This simpler drawing can be recovered from the final drawing by removing the vertices added in the Kleetope process. Using this observation, we build up a sequence of stronger properties for the biplanar and split thickness two embeddings of these graphs, as the number of Kleetope iterations increases. Eventually, these properties will become so strong that they lead to an impossibility.

Definition 1 *For a vertex v of graph G, it is convenient to denote the two copies of v in $2G$ by v_0 and v_1. We distinguish these from the two* images *of v_0 and the two images of v_1 in a biplanar or split thickness two drawing of $2G$. In such a drawing, v itself has four images, two from v_0 and two from v_1.*

We need the following additional definitions in the proof of our first lemma.

Definition 2 *Define the* excess *of a face in a planar, biplanar, or split thickness two drawing of a graph to be the number of edges in the face, minus three, so triangles have excess zero and all other faces have positive excess. Define the total excess of the drawing to be the sum of all face excesses.*

The total excess of a drawing equals the amount by which the number of edges in the graph falls short of the maximum possible number of edges in a drawing of its type, and so can be calculated only from a graph and the type of its drawing, independent of how it is drawn:

- A planar drawing of an n-vertex graph can have at most $3n-6$ edges. If there are m edges then the total excess is $(3n-6)-m$.
- A biplanar drawing of an n-vertex graph can have at most $6n-12$ edges ($3n-6$ in each planar subgraph). If there are m edges then the total excess is $(6n-12)-m$.
- A split thickness two drawing of an n-vertex graph can have at most $6n-6$ edges (obtained by combining the $3n-6$ bound on planar graphs with the $2n$ copies of vertices in the drawing). If there are m edges then the total excess is $(6n-6)-m$.

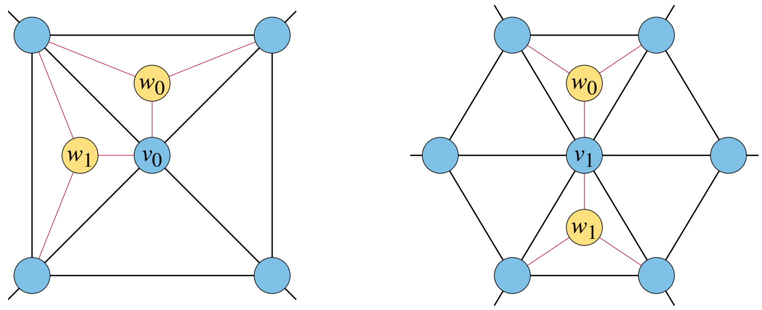

Figure 2: Left: Illustration for Section 2: If v in G has triangulated neighborhoods, and w is any neighbor of v added in KG, then the images of w must lie in two triangles incident to images of v, connected to all six triangle vertices. In this example, the four images of w are neighbors of only two images of v, but they may instead be neighbors of three or four images of v.

Lemma 1 *Let G be a maximal planar graph with n vertices. If $n \geq 49$, then in any biplanar drawing D of $2G$ some vertex v of G has images that are only incident to triangles. If $n \geq 73$, then for any split thickness two drawing some vertex v has the same property. We say that v has* triangulated neighborhoods*.*

Proof: A face with positive excess x has $x+3$ faces. Therefore, for a given total excess, the number of vertices that belong to non-triangular faces is maximized by distributing one unit of excess per face, to maximize the number of $+3$ counts coming from faces with positive excess. That is, regardless of the type of drawing (planar, biplanar, or split thickness two), the number of vertices that belong to non-triangular faces for a given excess x is maximized when there are x quadrilateral faces in the drawing, all having disjoint sets of vertices. In this case, the number of such vertices is $4x$. Any other distribution of the total excess, or non-disjointness among the non-triangular faces, produces fewer such vertices.

Because G is maximal planar, it has excess zero and $3n-6$ edges. Its blowup $2G$ has $2n$ vertices and $12n-24$ edges, four copies of each edge in G. Therefore, any biplanar drawing of $2G$ has excess 12, and any split thickness two drawing of G has excess 18. The number of vertices that can belong to a non-triangular face in these drawings is, respectively, 48 and 72. For values of n that are larger than this bound, some vertex v does not belong to any non-triangular face of the drawing. This vertex v necessarily has triangulated neighborhoods. □

Lemma 2 *Let G be a maximal planar graph, and consider any biplanar drawing or split thickness two drawing D of $2KG$, and the restriction of the same drawing to G. If some vertex v of G has triangulated neighborhoods in the restriction to G, then any neighbor w of v in $KG \setminus G$ has its four images each drawn surrounded by exactly three triangular faces. We say that w has* triangular neighborhoods*. If G has n vertices with $n \geq 49$, then in any biplanar drawing D of $2KG$ some vertex w of KG has triangular neighborhoods. If $n \geq 73$, then for any split thickness two drawing of $2KG$ some vertex w has triangular neighborhoods.*

Proof: As an added vertex in KG, w has degree three, so its copy w_0 in $2KG$ has degree six. To be adjacent to both v_0 and v_1, the two images of w_0 must lie in two triangular faces of the restricted drawing containing v_0 and v_1, as shown in Fig. 2. (No face contains both v_0 and v_1,

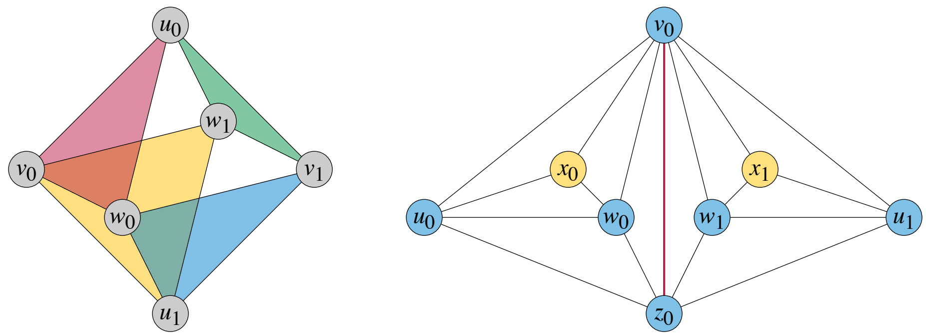

Figure 3: Left: Illustration for Section 2: partition of 2Δ into four triangles, for a triangle $\Delta = uvw$. Right: Illustration for Section 2: For the restriction of a given drawing to $2KG$, and for w in KG, images w_0 and w_1 have triangular neighborhoods sharing edge v_0z_0 (red). The third vertices of these triangular neighborhoods, u_0 and u_1, are distinct images of a neighbor u of w. For vertex x in K^2G adjacent to w and to u, two images have triangular neighborhoods without shared edges.

because they have triangular neighborhoods and are not adjacent.) This placement limits w_0 to having as neighbors only the six vertices of these two triangles, matching its degree, so it must be connected to all six of these vertices. The same argument applies to w_1.

The existence of w in biplanar or split thickness drawings of $2KG$ for graphs with many vertices follows by applying this argument to the vertex v with triangulated neighborhoods given by Section 2. □

In Section 2, the two triangular neighborhoods of w_0 must be disjoint, so they cover all six distinct neighbors of w_0 in $2KG$. Similarly, the two neighborhoods of w_1 must be disjoint. However, a neighborhood of w_0 may share a vertex or an edge with a neighborhood of w_1. (It cannot share edges with two neighborhoods because then those two neighborhoods would not be disjoint.) In fact, some sharing is necessary:

Lemma 3 *Let t be a vertex of a planar graph H having three neighbors, all adjacent. Suppose that t has triangular neighborhoods in a biplanar or split thickness two drawing of $2H$. Then it is impossible for all four images of t to have edge-disjoint neighborhoods in this drawing.*

Proof: Let Δ be the triangle of neighbors of t in H. 2Δ is isomorphic to $K_{2,2,2}$, the graph of a regular octahedron. We are assuming our drawings have no repeated edges, so two images of t with edge-disjoint neighborhoods in the drawing must come from edge-disjoint triangles of 2Δ. Thus, if the four images of t could be drawn with edge-disjoint triangular neighborhoods, these neighborhoods would form four edge-disjoint triangles of 2Δ. But in any subdivision of 2Δ into four edge-disjoint triangles (Fig. 3, left), each two triangles share a vertex, and together cover only five vertices of 2Δ. If the four images of t were placed in images of these four triangles, the two triangular neighborhoods of t_0 would miss one of the six neighbors of t_0 in $2H$, as would the triangular neighborhoods of t_1, preventing the drawing from being valid. Therefore, no such drawing is possible. □

Theorem 1 *Let G be a maximal planar graph with n vertices. If $n \geq 49$, then $2K^3G$ has no biplanar drawing, and if $n \geq 73$, then $2K^3G$ has no split thickness two drawing.*

Proof: Suppose for a contradiction that such a drawing existed, and consider the drawings within it of $2K^2G$ and of $2KG$. We will find a vertex with triangular neighborhoods in each of these drawings so that the four neighborhoods of the four images of the chosen vertex are nearly disjoint: these neighborhoods can share at most two edges total in $2KG$, at most one edge in $2K^2G$, and no edges in $2K^3G$. The existence of a vertex in K^3G whose triangular neighborhoods share no edges will contradict Section 2, showing that no such drawing can exist. To do this, we consider each level of iteration successively, as follows:

- In the drawing of $2KG$, Section 2 gives us a vertex w of KG with triangular neighborhoods. The neighborhood of each image of w shares at most one edge with other neighborhoods of images of w. For biplanar drawings this is immediate (only one other image is in the same planar subgraph and can share an edge with it). For split thickness two drawings, a neighborhood of an image of w_0 that shares edges with the neighborhoods of both images of w_1 is impossible, because then the two images of w_1 would share a vertex, preventing them from covering all six neighbors of w_1. Therefore, among the neighborhoods of all four images of w, there are at most two shared edges.

- To find a vertex x in K^2G with triangular neighborhoods in the drawing of $2K^2G$, with at most one edge shared among the neighborhoods of its four images, consider the vertex w found above in KG. If the neighborhoods of w have at most one shared edge in the drawing of $2KG$, let x be any neighbor of w added in forming K^2G from KG. Then x must again have triangular neighborhoods by Section 2. Because these triangular neighborhoods must be interior to the triangular neighborhoods of w, they can have at most one shared edge (the same edge as the one shared by the neighborhoods of w).

  Suppose, on the other hand, that the triangular neighborhoods of w share exactly two edges. Let Δ_0, Δ_0', Δ_1, and Δ_1' be the four triangles in $2KG$ neighboring the two images of w_0 and w_1, respectively, with an edge shared by Δ_0 and Δ_1 and another edge shared by Δ_0' and Δ_1'. Because these triangles can only share one edge, and each image of w must be adjacent to images of all three neighbors of w, the vertices of Δ_0 and Δ_1 that are not on the shared edge must be distinct images of the same vertex u. Choose a vertex x of $K^2G \setminus KG$, adjacent to w and to u.

  Because w has triangular neighborhoods in KG, its four images in the drawing of $2KG$ are each surrounded by three triangles, formed by images of w and two of its neighbors. For each image, only one of these triangles consists of the three neighbors of x. Thus, the four images of x must be placed in these four triangles. Two of these four triangles are subdivisions of Δ_0 and Δ_1, containing the non-shared images of u, and therefore do not share any edge with each other. The only possible shared edge among the triangular neighborhoods of x is the edge shared by Δ_0', and Δ_1', within which lie the other two images of x. Thus, by choosing x in K^2G we have eliminated one shared edge between triangular neighborhoods. See Fig. 3, right.

- If the four triangular neighborhoods of x in the drawing of $2K^2G$ are edge-disjoint, we already have a contradiction with Section 2. Otherwise, we must find a vertex t in K^3G whose triangular neighborhoods are edge-disjoint, giving us the desired contradiction. To do so, consider the four triangular neighborhoods of x in the drawing of $2K^2G$, only two of which share one edge. For the two triangles that share an edge, the two non-shared vertices of these triangles must be distinct images of the same vertex y in K^2G. Choose a vertex t of

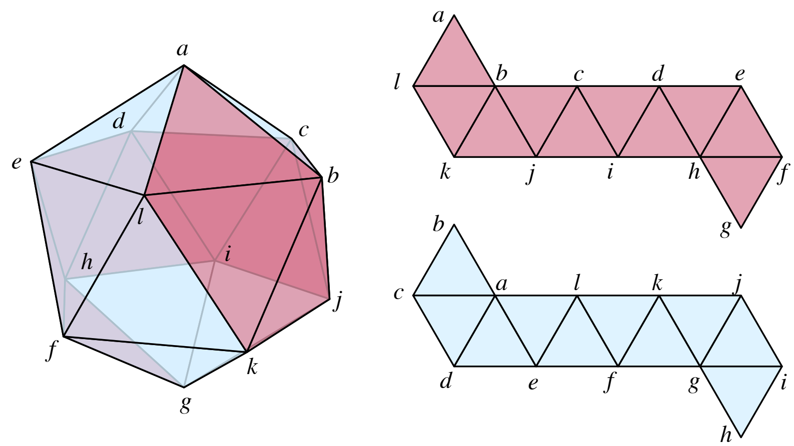

Figure 4: Decomposition of an icosahedron into two outerpaths.

$K^3G \setminus K^2G$, adjacent to x and to y. Then the four triangles in which t must be placed lie within the four triangular neighborhoods of x, away from the shared edge of these triangular neighborhoods, so they cannot share any edges with each other. By Section 2, this is an impossibility.

□

When G is not maximal planar or is too small for the theorem to apply directly, more iterations of the Kleetope operation can be used before applying the same argument. Thus, there exists an i such that for every plane graph G, $2K^iG$ is not biplanar and has no split thickness two drawing.

3 Drawings from Triangle Strips

Definition 3 *An* outerpath *is an outerplanar graph whose weak dual (the adjacency graph of its bounded faces) is a path.*

Suppose that the dual vertices of a planar graph G can be partitioned into two subsets that each induce a path in the dual graph. Then the dual cut edges between these two subsets correspond, in G itself, to a Hamiltonian cycle that partitions G into two outerpaths. Fig. 4 depicts an example of this sort of *two-outerpath decomposition* for the graph of the regular icosahedron.

Theorem 2 *If a planar graph G has a two-outerpath decomposition, then $2G$ has a biplanar drawing.*

Proof: We may assume without loss of generality, by triangulating each outerplanar graph if necessary, that both outerpaths are maximal outerplanar: each of their faces is a triangle. If this triangulation step adds two copies of the same edge to the graph, it is not a problem, because the edges added in this triangulation step will be removed from the final drawing.

In each outerpath, the triangular faces form a linear sequence, separated by the internal edges of the outerpath, which are also linearly ordered. In the blowup $2G$, number the two copies of each vertex v as v_0 and v_1. If uv is any edge of G, then $2uv$ is a four-vertex cycle $u_0v_0u_1v_1$.

We will construct a biplanar drawing of $2G$ with each plane containing all copies of interior edges of one of the two outerpaths, and two out of the four copies of each boundary edge. We draw copies of the interior edges as nested quadrilaterals, one for each diagonal of its outerpath, in the same order that these diagonals appear within the outerpath. If we draw these quadrilaterals one at a time, from the innermost to the outermost, then each two consecutive quadrilaterals share two opposite vertices, corresponding to the single shared endpoint of the two diagonals.

In each pair of consecutive quadrilaterals, the outer quadrilateral has two potential orientations with respect to the inner one: if the two consecutive diagonals are uv and vw, with quadrilateral $u_0v_0u_1v_1$ drawn inside quadrilateral $v_0w_0v_1w_1$, then these quadrilaterals may be drawn so that pairs u_0w_0 and u_1w_1 are adjacent, or so that pairs u_0w_1 and u_1w_0 are adjacent. In one plane we always choose the orientation with u_0w_0 and u_1w_1 adjacent, and we connect these pairs of vertices by an edge. In the other plane, we always choose the orientation with u_0w_1 and u_1w_0 adjacent, and we connect these pairs of vertices by an edge. In this way, we draw all four copies of each boundary edge, except the edges incident to the two ears (triangles with two boundary edges).

It remains to draw the ears. Each has two boundary edges sharing a vertex, which we call the *ear vertex*. These two boundary edges have not yet been drawn in the plane of their outerpath. The third edge of the ear is a diagonal whose images form the innermost or outermost quadrilateral in its plane. We place both copies of the ear vertex inside or outside this quadrilateral (respectively as it is innermost or outermost), connected to its two neighbors in the ear with the same numbering convention: in the plane where the quadrilaterals are oriented with u_0w_0 and u_1w_1 adjacent, we connect each ear vertex to neighbors with the same subscript, and in the other plane we connect each ear vertex to neighbors with the opposite subscript.

Thus, all copies of the diagonals of one strip and all copies of boundary edges that connect copies having the same index are drawn in one plane. All copies of the diagonals of the other strip and all copies of boundary edges that connect copies having different indices are drawn in the other plane. The result is a biplanar drawing of the entire blowup $2G$. □

Fig. 5 depicts the drawing obtained by applying Section 3 to the outerpath decomposition of the icosahedron depicted in Fig. 4. The following extension of this result is noteworthy in connection with our iterated Kleetope counterexample:

Theorem 3 *If a maximal planar graph G has a decomposition into two outerplanar graphs, coming from an induced path partition of its dual graph into two paths, then $2KG$ has a drawing with split thickness two.*

Proof: We construct the drawing of Section 3 for $2G$, place the two planes of this biplanar drawing side-by-side in a single plane (as depicted in Fig. 5), and augment the resulting split drawing to include the additional vertices of $2KG$. For each triangular face Δ of G, one added vertex v_Δ of KG is adjacent to the vertices of Δ. In $2KG$, there are two copies of v_Δ, each of which should be drawn (in two images) adjacent to all six copies of vertices of Δ.

If Δ is not an endpoint of the dual path to which it belongs, the drawing of $2G$ will include four images of Δ, in two disjoint pairs. For instance, in Fig. 5, the four images of triangle *blk* can be seen near the outer boundary of the left part of the drawing. For each copy of v_Δ, its two images can be placed into two disjoint images of Δ, which together include images of all six copies of vertices of Δ.

If Δ is an endpoint of the dual path to which it belongs, the drawing of $2G$ will include two disjoint triangular faces and one hexagonal face formed from the vertices of Δ. For instance, in Fig. 5, the images of the vertices in triangle *abc* form the outer hexagonal face and two triangular

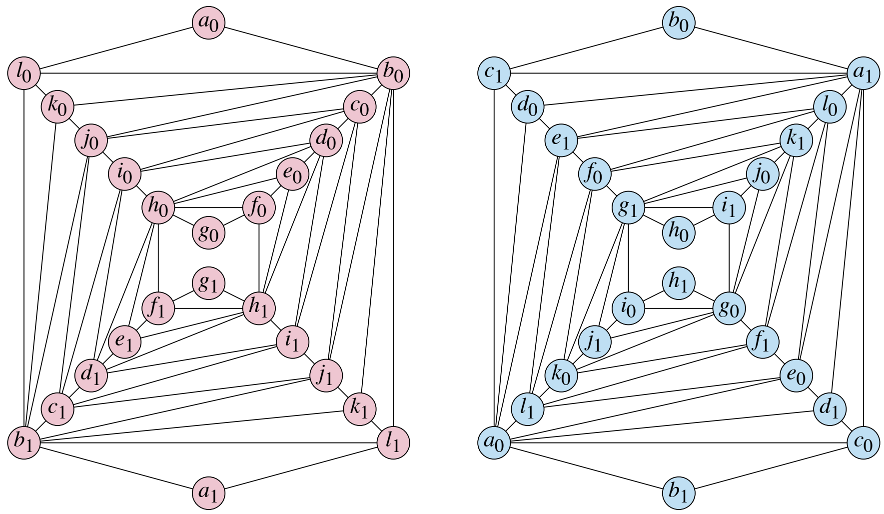

Figure 5: Biplanar drawing of the 2-blowup of an icosahedron corresponding to the outerpath decomposition of Fig. 4.

faces of the right part of the drawing. The triangles are each adjacent to the hexagon but not to each other. In this case, one copy of v_Δ may be drawn with two images in the two triangles, while the other copy may be drawn with a single image in the hexagon. □

For instance, the triakis icosahedron, the Kleetope of the icosahedron, is a maximal planar graph whose edges all have total degree ≥ 13. (This is the maximum possible for the minimum total degree of an edge, by Kotzig's theorem [21].) Because the icosahedron has a two-outerpath decomposition (Fig. 4), we can apply Section 3 to its Kleetope, producing a drawing with split thickness two of the 2-blowup of the triakis icosahedron.

In general, we cannot extend this construction to higher-order Kleetopes. When a graph G has a two-outerpath decomposition, the drawings of $2KG$ produced from G by Section 3 again have four images of each triangular face of KG, but some of these quadruples of images cannot be grouped into disjoint pairs. However, in the method of Section 3 each added vertex of K^2G corresponds to two vertices in $2K^2G$, and the two images of each of these two vertices must be placed in disjoint triangles, in order to provide all six of its adjacencies. Therefore, this method does not provide drawings of $2K^2G$. However, in one special case, for $G = K_4$ (the graph of a tetrahedron), a different method works. In this case, the Kleetope KK_4 has an outerpath decomposition, shown in Fig. 6. Therefore, applying Section 3 we can obtain a split thickness two drawing of $2K^2K_4$.

A very similar drawing algorithm to the one in Section 3 can be used for graphs with a different form of decomposition into triangle strips.

Definition 4 *Let G be a planar graph having both a Hamiltonian path P and a dual Hamiltonian path P^*, with no edge and its dual edge belonging to both paths. Then we call (P, P^*) a* path–copath decomposition. *It is a special case of the* tree–cotree decomposition *formed from any spanning tree of a planar graph and the dual spanning tree formed by the duals of the complementary set of edges [9].*

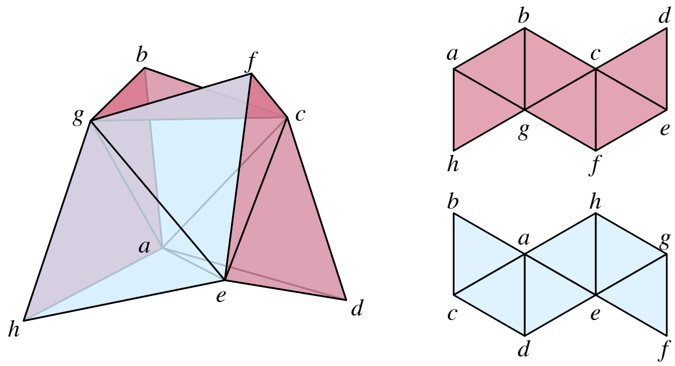

Figure 6: Outerpath decomposition of KK_4.

Theorem 4 *Let planar graph G have a path–copath decomposition. Then $2G$ has a drawing with split thickness two.*

Proof: We may triangulate the faces of G, if necessary, preserving the existence of a path–copath decomposition by choosing added diagonals that split each face into an outerpath. As a result, the dual path P^* of the path–copath decomposition (P, P^*) becomes a triangulated outerpath. The outerpath can be formed from a plane drawing of G by cutting the plane along each edge of P, causing each edge of P to appear exactly twice on the boundary of the outerpath. Each vertex of G may appear multiple times on the boundary of this outerpath, with multiplicity equal to its degree in P (at most two, because P is a path).

We apply the method of Section 3 to this single outerpath, producing a drawing in which each appearance of a vertex v of G on the boundary of the outerpath produces images of both copies of v. If v appears once on the boundary of the outerpath, its two copies each appear once. If v appears twice, its two copies appear twice, giving this drawing split thickness two. This drawing style automatically produces all four images of each edge interior to the outerpath. For an edge uv of path P, appearing twice on the boundary of the outerpath, we choose arbitrarily which appearance of uv on the boundary of the outerpath is used to draw edges u_0v_0 and u_1v_1, and which is used to draw edges u_0v_1 and u_1v_0. In this way, all four images of uv are drawn correctly. □

Fig. 7 depicts an example.

4 Drawings from Colorings

We show in this section that the blowup $2G$ of a 3-colored planar graph G has split thickness at most two. We do not know whether all such blowups are biplanar; our construction does not produce a biplanar drawing. More generally, we show that kG has split thickness at most k; the result for $2G$ is a special case.

Theorem 5 *Let G be planar and 3-chromatic; then kG has a drawing with split thickness k.*

Proof: Color the vertices of G red, blue, and yellow, and number the copies of each vertex in kG from 0 to $k-1$. Draw kG as k^2 disjoint copies of G, where for (i, j) with $0 \le i, j < k$ we draw a copy of G consisting of the copies of red vertices numbered i, the copies of blue vertices numbered j,

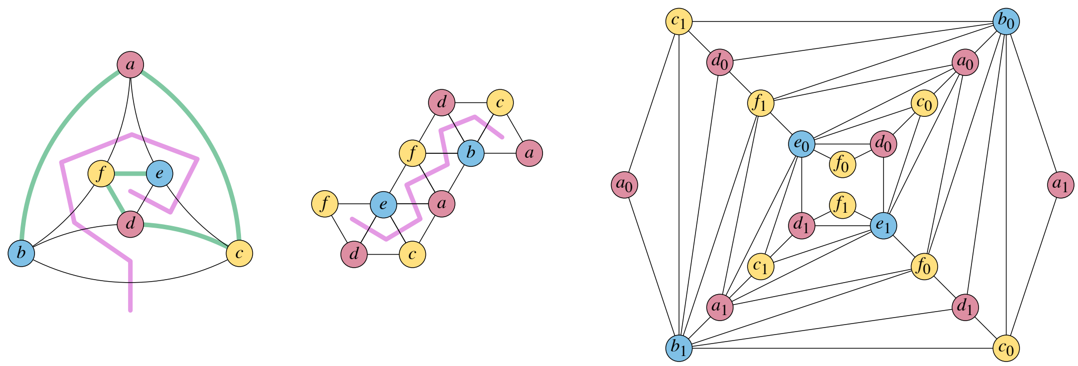

Figure 7: Path–copath decomposition of the octahedral graph $K_{2,2,2}$ (left), the outerpath obtained by cutting the path (center), and the split thickness 2 drawing of $2K_{2,2,2} = K_{4,4,4}$ obtained from Section 3 (right).

and the copies of yellow vertices numbered $-(i + j) \bmod k$. As the disjoint union of k^2 planar drawings, the result is planar. Each edge of kG appears in one copy of G, and each vertex in kG has images in k copies of G. As a planar drawing with k images of each vertex, it is a drawing with split thickness k. □

Fig. 8 shows a drawing of $K_{6,6,6}$ with split thickness three, obtained by applying this construction to the triple blowup of the graph of the octahedron. When applied to planar bipartite graphs, the same construction yields a thickness k drawing of the k-blowup. In this case, we may consider the two colors of a given planar bipartite graph to be red and blue, with no yellow vertices. The drawing of Section 4 produces k^2 copies of G. Number k planes from 0 to $k - 1$, and place k copies of G onto each of these planes, where a copy with red vertices numbered i and blue vertices numbered j is placed onto plane $-(i + j) \bmod k$. That is, each copy is placed onto a plane that has the same index as the yellow vertices in the copy would have, if there were any yellow vertices. With this numbering and placement, each copy of each vertex appears once in each plane. In the case $k = 2$, it is also possible to find biplanar drawings of the 2-blowups of planar bipartite graphs in a different way, using the fact that planar bipartite graphs have arboricity at most two.

5 Computational complexity

This work naturally raises several questions in computational complexity, concerning the time needed for testing biplanarity of blowups (of planar graphs or more generally), for testing the split thickness of blowups, and for testing the existence of finding two-outerpath decomposition and of path–copath decompositions. Biplanarity is NP-complete in general [24], but the proof does not apply to the special case of blowups. Similarly, although partition into two induced paths is NP-complete for general graphs [22], and its planar case is closely related to Hamiltonicity of the dual graph, we are unaware of complexity results for this case.

We provide the following partial results:

Theorem 6 *Testing whether the blowup kG of a given graph G has thickness or split thickness t is fixed-parameter tractable in the combination of two parameters: k and w, where w is the treewidth of G.*

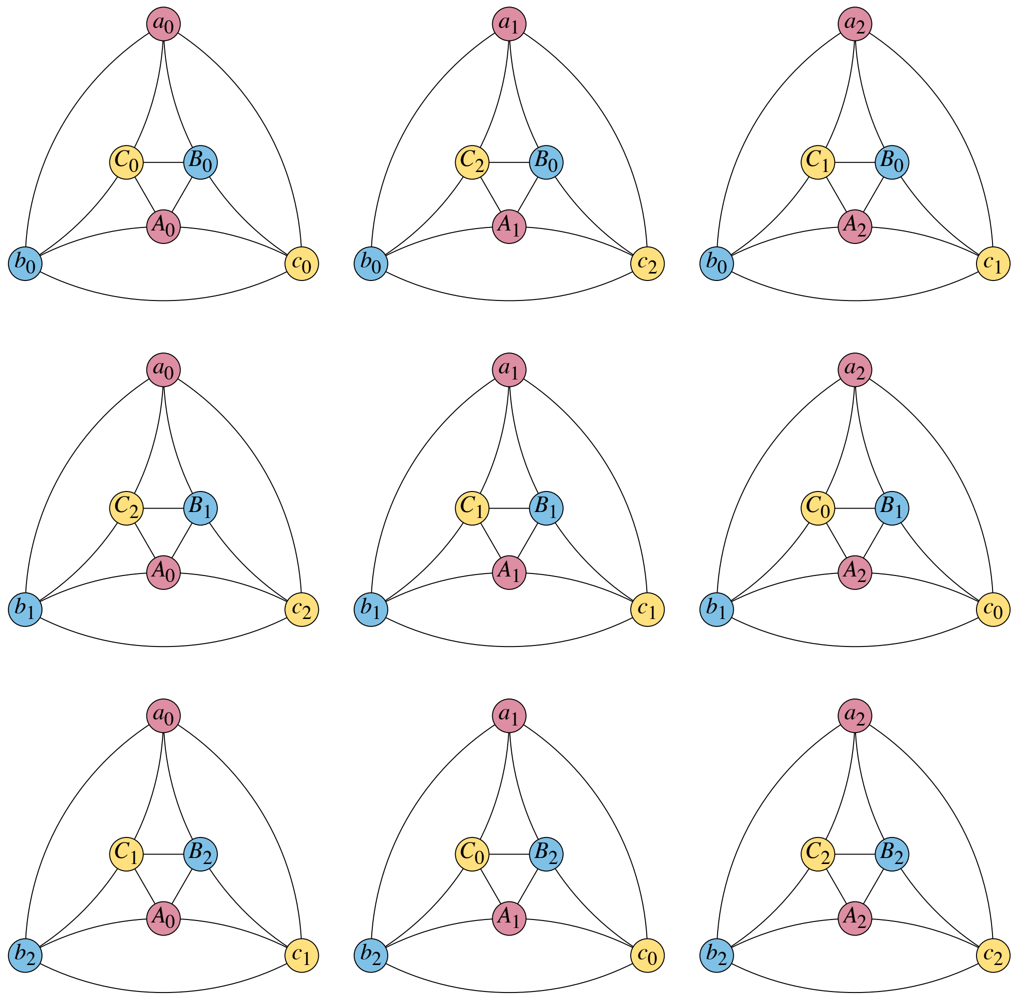

Figure 8: Section 4 applied to the graph $3K_{2,2,2} = K_{6,6,6}$. Each vertex is labeled with a letter (its position in $K_{2,2,2}$), a number (its index as a copy in the blowup), and a color in the coloring of $K_{2,2,2}$. Each letter-number combination has three images, so this is a drawing with split thickness three. $K_{6,6,6}$ has 108 edges, but drawings of 18-vertex graphs with split thickness two can have at most 102 edges, so this drawing is optimal.

Proof: If G has treewidth w, its blowup kG has treewidth at most $k(w+1)-1$, obtained from a tree decomposition of G by replacing each bag of the decomposition (a set of $\leq w+1$ vertices of G) with the set of copies of these vertices. The result follows from the known fixed-parameter tractability of thickness and split thickness in the treewidth [12], obtained by applying Courcelle's theorem on fixed-parameter tractability of properties described in second-order logic. Planarity can be formulated logically in terms of the non-existence of the forbidden minors K_5 and $K_{3,3}$, and the planarity of each of t subgraphs of a given graph or of a graph obtained by subdividing the vertices of a given graph into t copies can be formulated logically from the given graph itself using standard methods for expanding a logical formula to apply to graphs derived from a given graph.

The resulting logical formula has size proportional to t, and its satisfiability can be tested on graphs of width $k(w+1)-1$ in time that is fixed-parameter tractable in k, w, and t. The dependence of this time bound on t can be eliminated by observing that all graphs have thickness and split thickness at most equal to their treewidth. □

Theorem 7 *Testing whether a plane graph G has a two-outerpath decomposition or a path–copath decomposition is fixed-parameter tractable in the treewidth of G.*

Proof: Again, we apply Courcelle's theorem, using a logical characterization of these decompositions. For a two-outerpath decomposition, it is simplest to apply the theorem to the dual graph of G, using the known result that dualization can increase the treewidth by at most one [3]. In the dual, a two-outerpath decomposition becomes a partition of the dual vertices into two induced paths. This may be described logically by the existence of a set of vertices (the vertices in one of the paths) such that for both this set and its complement, all vertices have one or two neighbors, exactly two vertices have only one neighbor, and such that no partition of the set into two proper subsets has no edges spanning the partition.

For a path–copath decomposition, we replace G by its barycentric subdivision, a maximal planar graph that has a vertex for each vertex, edge, or face of G, and a triangle for each incident triple of a vertex, edge, and face, with each vertex of the barycentric subdivision labeled by the type of object in G that it comes from. Passing to the barycentric subdivision increases the treewidth to a function of its previous value; therefore, for graphs of bounded treewidth, the barycentric subdivision has bounded treewidth [4]. A path–copath decomposition then consists of two disjoint induced paths, described logically as sets of vertices as above, such that one path alternates between vertices and edges of G, the other path alternates between faces and edges of G, and both paths together span all the edges of G. □

Whether these problems are polynomial without parameterization, or whether they are NP-complete, remains open. The use of treewidth as a parameter is nontrivial for these problems, though. On the one hand, our proof of the existence of graphs whose blowup is not biplanar and does not have split thickness two can be restricted to planar 3-trees, the graphs obtained by gluing together tetrahedra face-to-face in the pattern of a tree. Indeed, the iterated Kleetope of a tetrahedron is a special case of a planar 3-tree. On the other hand, there exist maximal planar graphs with two-outerpath decompositions (and hence biplanar blowups) that contain arbitrarily large regular triangular grids and hence have arbitrarily high treewidth. The construction involves placing vertices at equal spacing along two helices on the same cylindrical surface, offset by half a phase from each other, triangulating the strip between the two helices (Fig. 9), and capping off the ends of the resulting cylindrical surface to make a maximal planar graph. The resulting triangulated surface has a two-outerpath decomposition by construction, and it contains triangular grids whose dimensions are the number of points per winding of the helices and the number of windings of the helices.

6 Conclusions

We have shown that 2-blowups of iterated Kleetopes are not biplanar, but that 2-blowups of planar graphs with outerpath decompositions are biplanar. Additionally, we have shown that 2-blowups of graphs with path–copath decompositions have split thickness at most 2, and k-blowups of planar graphs with chromatic number at most three have split thickness at most k.

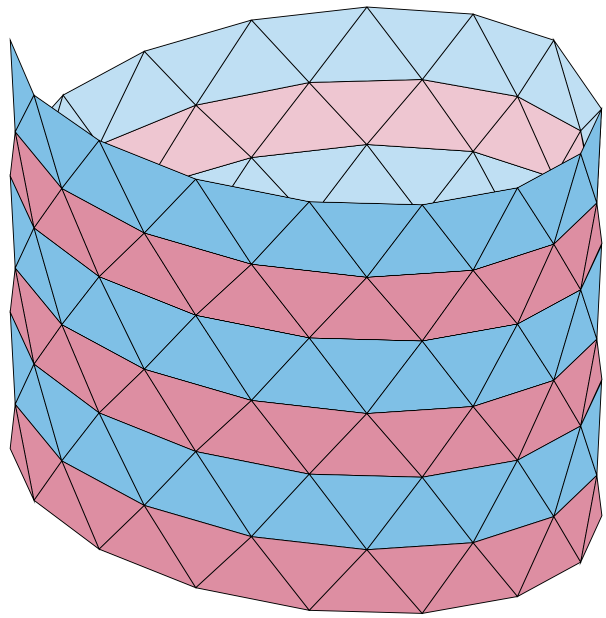

Figure 9: Part of a construction for a maximal planar graph with a two-outerpath decomposition (blue and pink) containing an arbitrarily large regular triangular grid.

Several natural questions remain open for future research:

- Is it ever possible for the 2-blowup of a 3-chromatic planar graph to be non-biplanar? Is it ever possible for the 2-blowup of a 4-vertex-connected planar graph to be non-biplanar?
- Do apex-outerplanar graphs [6] have biplanar blowups?
- Can Section 3, on drawing 2-blowups of graphs with a two-outerpath decomposition, be extended from thickness to geometric thickness? Geometric thickness (also called real linear thickness) is similar to thickness, but requires vertices to have the same geometric placement in each planar subgraph and requires edges to be drawn as non-crossing line segments [5, 7, 8, 10, 20].

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