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Drawing outer-1-planar graphs revisited

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Abstract. In a recent article (Auer et al., Algorithmica 2016) it was claimed that every outer-1-planar graph has a planar visibility representation of area $O(n \log n)$. In this paper, we show that this is wrong: There are outer-1-planar graphs that require $\Omega(n^2)$ area in any planar drawing. Then we give a construction (using crossings, but preserving a given outer-1-planar embedding) that results in an orthogonal box-drawing with $O(n \log n)$ area and at most two bends per edge.

1 Introduction

A *1-planar* graph is a graph that can be drawn in the plane such that every edge has at most one crossing. Many graph-theoretic and graph-drawing results are known for 1-planar graphs, see for example [12]. One subclass of 1-planar graphs is the class of *outer-1-planar (o1p) graphs*, which have a 1-planar drawing such that additionally every vertex is on the outer face (the unbounded region of the drawing).

Outer-1-planar graphs were introduced by Eggleton [10] and studied by many other researchers [1, 2, 8, 11]. Of particular interest here is a paper by Auer, Bachmeier, Brandenburg, Gleißner, Hanauer, Neuwirth and Reislhuber [2]. Among many results, they characterize the forbidden minors of outer-1-planar graphs, give a recognition algorithm, and give bounds on various graph parameters such as the number of edges, treewidth, stack number and queue number. Finally they turn to drawing algorithms for outer-1-planar graphs, and here claim the following result: "Every o1p graph has a planar visibility representation in $O(n \log n)$ area." (Theorem 8).

In this paper, we show that this result is incorrect. Specifically, we construct an n-vertex outer-1-planar graph such that in *any* planar embedding there are $\Omega(n)$ nested triangles (we give detailed definitions below). It is known [7] that any planar graph drawing with k nested cycles

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requires width and height at least $2k$ in any planar poly-line drawing. Since any planar visibility representation can be converted into a poly-line drawing of asymptotically the same width and height (see also Figure 3), any planar visibility representation of our graph uses $\Omega(n^2)$ area and the claim by Auer et al. is incorrect.

Then we give a drawing algorithm for outer-1-planar graphs that achieves area $o(n^2)$. The resulting drawings have crossings, but reflect exactly the given outer-1-planar embedding. Our construction gives orthogonal box-drawings with area $O(n \log n)$ and at most two bends per edge; they can be converted to poly-line drawings of the same area.

To our knowledge, the only prior result on orthogonal drawings of outer-1-planar drawings (other than the one by Auer et al. that we disprove) is by Argyriou et al. [1]: they showed that every outer-1-planar graph with maximum degree 4 has a point-orthogonal drawing with $O(n^2)$ area and at most 2 bends per edge such that the given outer-1-planar embedding is respected. In follow-up to the current work, the author (with co-authors) also studied embedding-preserving visibility representations of outer-1-planar graphs; for these area $O(n^{1.5})$ can always be achieved and this is tight for some graphs [6]. The same paper also proves that outer-1-planar graphs have so-called bar-1-visibility representations of area $O(n \log n)$, albeit not embedding-preserving.

2 Definitions

We assume familiarity with graphs, see e.g. [9]. A *planar graph* is a graph that can be drawn in the plane without any crossing. Such a *planar drawing* Γ defines the *cells*, which are the connected regions of $\mathbb{R}^2 \setminus \Gamma$. [1] The infinite cell is called the *outer face*, the other cells are called *inner cells*. A planar drawing defines the *planar embedding* consisting of the *rotation scheme* (the clockwise order of edges at each vertex) and the outer face (described via the circuit that bounds it). A graph is called *outer-planar* if it has a planar embedding where all vertices are on the outer face.

A *1-planar graph* is a graph that can be drawn in the plane such that every edge has at most one crossing. As above, such a *1-planar drawing* defines *cells* and the *outer face*. An *outer-1-planar* graph is a graph with a 1-planar drawing where additionally all vertices are on the outer face. Any such drawing is described via an *outer-1-planar embedding*, consisting of the rotation scheme, the outer face, and information as to which pair of edges cross.

A graph is called *maximal* (within a graph class) if no edge can be added while staying in the same graph class and having no duplicate edges or loops. In a maximal outer-planar graph the outer face is a cycle of length n and all inner faces are triangles. In a maximal outer-1-planar graph the outer face is a cycle of length n where no edges have crossings [10]. We only draw graphs that are maximal (within their class); one can always make a graph maximal by adding edges, and delete those edges from the obtained drawing later.

A *poly-line drawing* of a graph is a drawing where vertices are points and edges are polygonal curves; a *bend* is the transition-point between segments of the polygonal curve. An *orthogonal point-drawing* is a poly-line drawing where edge-curves are *orthogonal*, i.e., use only only horizontal and vertical segments. We also consider *orthogonal box-drawings*, where vertices are represented by axis-aligned boxes and edges are orthogonal curves. A special kind of orthogonal box-drawing is a *visibility representation* where edges have no bends. In the orthogonal box-drawings studied in this paper the vertices are *flat*: they are represented by horizontal line segments (in the figures, we show them thickened into a thin rectangle). We call such a vertex-box a *bar* and such an orthogonal

[1] For planar drawings the term "face" is more common, but we use "cells" here since we soon also study drawings with crossings.

box-drawing an *orthogonal bar-drawing*.

We assume (without further mentioning) that all our drawings are *grid-drawings*, i.e., all defining features (vertex-points, endpoints of vertex-bars, bends) are placed at points with integer coordinates. A *row* [*column*] is a horizontal [vertical] grid-line that intersects the smallest axis-aligned rectangle that encloses the drawing. The *height* [*width*] of a drawing is the number of rows [columns]; the area is the width times the height. We call a drawing *embedding-preserving* if it exactly reflects a given (planar or 1-planar) embedding of the graph.

In what follows, we usually identify the graph-theoretic object (vertex, edge) with the geometric object (bar, point, poly-line) used to represent it, and so for example say "s occupies the top right corner" rather than "the bar of s occupies the top right corner".

3 Lower bound

In this section, we construct an outer-1-planar graph that requires $\Omega(n^2)$ area in any planar poly-line drawing. For this we need a graph G_L (for $L \geq 2$ even) that consists of a $2 \times L$-grid with a crossing in every second inner cell. Clearly this is an outer-1-planar graph, see Figure 1. Enumerate the vertices of G_L as in the figure.

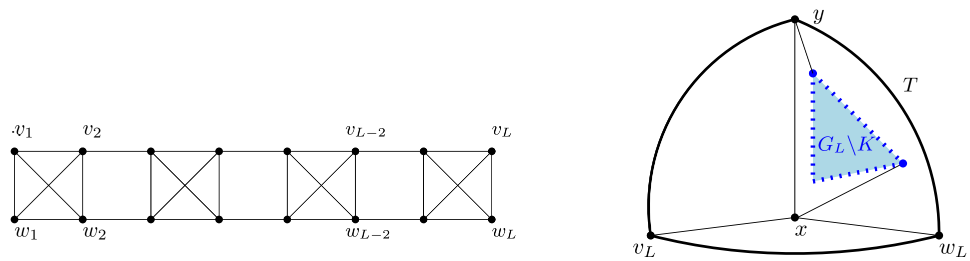


Figure 1: The outer-1-planar graph G_8, and how to find nested triangles.

It is known that all outer-1-planar graphs are planar [2], but they can have many different planar embeddings. However, we can show that for G_L, all planar embeddings are bad in some sense.

Call a set of disjoint triangles $T_1, \ldots, T_\ell$ *nested* (in a fixed planar embedding) if for $i = 2, \ldots, \ell$ the vertices on the outer-face of the embedding induced by $T_1 \cup \cdots \cup T_i$ are those of triangle T_i.

Lemma 1 *Fix $L \geq 2$ even. Any planar embedding Γ_L of G_L with $\{v_L, w_L\}$ on the outer face contains $L/2$ nested triangles.*

Proof: Set $K := \{v_L, w_L, v_{L-1}, w_{L-1}\}$. These four vertices form a K_4; its induced planar embedding Γ_K is hence unique up to renaming. By assumption the outer face T of Γ_K contains v_L, w_L and one vertex $y \in \{v_{L-1}, w_{L-1}\}$; set $x = \{v_{L-1}, w_{L-1}\} \setminus y$.

If $L = 2$, then we are done (use triangle T). If $L > 2$, then graph $G' := G_L \setminus K$ is connected, so must reside entirely within one cell f of Γ_K. Graph G' contains neighbours of x and y, so cell f must contain both x and y. Since x is not on the outer face of Γ_K, cell f is not the outer face of Γ_K. So no vertex of $G_L \setminus K$ is within the infinite region of Γ_K, making T the outer face of the entire drawing Γ_L.

Observe that $G_L \setminus K$ is a copy of G_{L-2}. Since both v_{L-2} and w_{L-2} have neighbours in $\{x, y\}$, vertices $\{v_{L-2}, w_{L-2}\}$ are on the outer face of the induced drawing Γ_{L-2} of $G_L \setminus K$. By induction,

Γ_{L-2} contains $L/2-1$ nested triangles $T_1,\dots,T_{L/2-1}$. Adding the outer face T to this gives the desired set of nested triangles for G since G_{L-2} resides within the region bounded by T. □

We now give two lower bounds. The second one is slightly weaker, but it holds even for *IC-planar graphs*, i.e., 1-planar graphs where no two crossings have an endpoint in common. Furthermore, the graph has maximum degree 4, so the lower bound holds even for planar orthogonal point-drawings.

Theorem 1 *Fix an arbitrary integer $N \geq 1$.*

(a) There exists an outer-1-planar graph with $n \geq N$ vertices that requires width and height at least $(n+2)/4$ in any planar poly-line grid-drawing.

(b) There exists an IC-planar outer-1-planar graph with $n \geq N$ vertices and maximum degree 4 that requires width and height at least $n/4$ in any planar poly-line grid-drawing.

Proof: Set $L = 2N$. Take two copies of G_L; we use primed names (i.e., G'_L, v'_L, w'_L) for the second copy and its vertices. For (a), let G_a be the graph obtained by identifying v_L with v'_L and identifying w_L with w'_L. For (b), let G_b be the graph obtained by inserting edges (v_L, v'_L) and (w_L, w'_L). See Figure 2. Clearly the graphs are outer-1-planar, and G_b is IC-planar and has maximum degree 4.

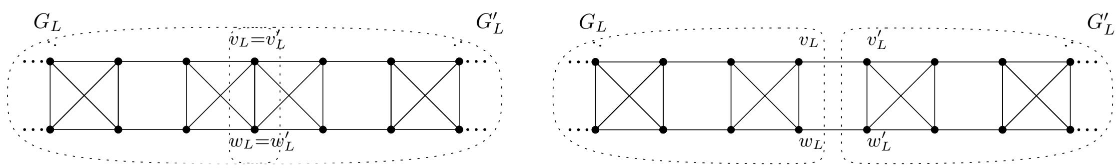


Figure 2: Using G_L to find lower-bound graphs G_a and G_b.

Fix an arbitrary drawing Γ_a of G_a, and consider the induced drawings Γ_L and Γ'_L of G_L and G'_L. In at least one of Γ_L and Γ'_L, edge (v_L, w_L) must be on the outer-face of the induced drawing. Up to renaming, drawing Γ_L hence has $L/2$ nested triangles by Lemma 1. It is known [7] that ℓ nested triangles require width and height 2ℓ in any planar poly-line drawing, which implies (a) by $n = 4L-2$.

The proof for G_b is almost exactly the same. Fix a drawing of G_b and let Γ_L, Γ'_L be the induced drawings of G_L and G'_L. Either Γ_L has (v_L, w_L) on the outer-face or Γ'_L has (v'_L, w'_L) on the outer-face, so one of them contains $L/2$ nested triangles and hence requires width and height $L = n/4$. □

Thus Theorem 1 shows that the area-bound of $O(n \log n)$ for a planar visibility representation claimed by Auer et al. [2] is not correct. Readers familiar with [2] might be curious to know where exactly their approach went wrong. We will discuss this briefly later (in Section 4.6) since the tools from our own construction will be useful for explaining how their construction works.

4 Constructions

We now turn towards constructing orthogonal bar-drawings of outer-1-planar graphs with $O(n \log n)$ area and $O(1)$ bends per edge. This is very easy to achieve if we do not care about the constants in the O-bounds.

Observation 2 *Any outer-1-planar graph has an embedding-preserving orthogonal bar-drawing with $O(1)$ bends per edge and $O(n \log n)$ area.*

Proof: After adding edges, we may assume that the input graph G is maximal outer-1-planar. Remove all edges that are crossed; the resulting *skeleton* S is outer-planar and all inner cells contain three or four vertices [8].

S has an embedding-preserving planar orthogonal bar-drawing Γ_S with area $O(n \log n)$ and at most two bends per edge [3, 4]. To turn this into a drawing of G, we need to re-insert the edges that had crossings. To have space to do this, insert first four new grid-lines each before and after each grid-line of Γ_S; the area remains $O(n \log n)$.

Let $e = (u, v)$ and $e' = (w, x)$ be two edges that crossed each other in G, and let f be the inner cell of S that results from removing e, e'. The boundary of f has four vertices and four edges since G is maximal outer-1-planar. Each edge has at most two bends in Γ_S. So the boundary of f in Γ_S has $O(1)$ bends. Pick a grid-point c inside f, and connect it via disjoint orthogonal paths inside f to u, v, w, x; this then draws e and e' with a crossing at c. Each orthogonal path requires only $O(1)$ bends and can be drawn along grid-lines, since we can trace along the boundary of f and use the four new grid-lines inserted next to it. Repeating for all pairs of crossing edges gives the desired drawing of G. □

We now work on reducing the constants in the O-bounds and show:

Theorem 3 *Any outer-1-planar graph has an embedding-preserving orthogonal bar-drawing with at most two bends per edge and $O(n \log n)$ area.*

It is straight-forward to convert any orthogonal bar-drawing into a poly-line drawing while keeping the width the same and at most tripling the height, see also Figure 3. Every edge receives at most one more bend at each endpoint. (Even better conversion-results are known for planar drawings [5], but it is not clear whether those also apply to 1-planar drawings.) Therefore Theorem 3 implies:

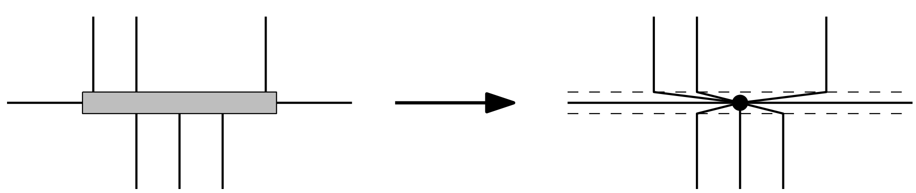


Figure 3: Converting a bar-drawing into a poly-line drawing.

Corollary 4 *Any outer-1-planar graph has an embedding-preserving poly-line drawing with at most four bends per edge and $O(n \log n)$ area.*

4.1 Drawing types

Now we prove Theorem 3 with a recursive drawing algorithm. Since we have a constant number of bends per edge, and any outer-1-planar graph has $O(n)$ edges [2], we have $O(n)$ vertical segments in the orthogonal bar-drawing of Theorem 3. As such, after deleting empty columns if needed, the width is automatically in $O(n)$. It will be immediately obvious from the figures illustrating the constructions that every edge is drawn with at most 2 bends. Therefore the analysis of our algorithm is focused on the height of the drawing, which we prove to be in $O(\log n)$.

Our drawing algorithm borrows many ideas from our previous result on how to draw outer-planar graphs (see [3] for the original version and [4] for a different and more detailed exposition). We need some notations. Throughout, G denotes an outer-1-planar graph with a fixed embedding. After adding edges we may assume that G is maximal outer-1-planar. We define the *size* $|G|$ of G to be $n-1$, i.e., one less than the number of vertices; this may be rather unusual but will help keep later equations simpler. Fix a *reference-edge* (s,t) on the outer face of G. The endpoints of (s,t) are called the *poles*, and (after possible renaming) we assume that s comes before t in clockwise order along the outer-face.

The crucial idea in [3, 4] was to fix the location of the poles in the drawing. We use the same idea, but allow more types of drawings (see also Figure 4) to achieve fewer bends overall. An orthogonal bar-drawing Γ of G is called

- a drawing of *type* A if s and t occupy the top right and bottom right corner of Γ, respectively (this is the drawing type that was used in [3, 4]);
- a drawing of *type* $\overline{B}$ if s occupies the top right corner of Γ, and t occupies the point one row below this corner;
- a drawing of *type* $\underline{B}$ if t occupies the bottom right corner of Γ, and s occupies the point one row above this corner;
- a drawing of *type* C if s and t occupy the bottom left and bottom right corner of Γ, respectively.

All drawings that we create are embedding-preserving. In particular edge (s,t) must be drawn clockwise along the boundary of the drawing; Figure 4 shows how it will typically be drawn.

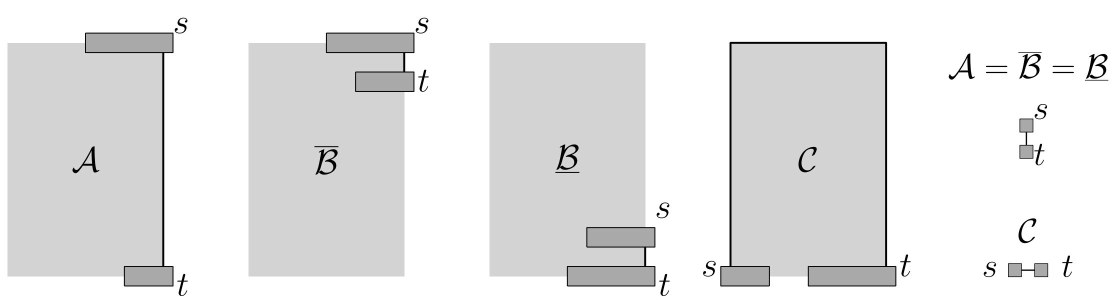


Figure 4: The drawing-types, and the base cases.

Let $\alpha \approx 0.59$ be such that $\alpha^5 = (1-\alpha)^3$. Let $\phi := \frac{\sqrt{5}-1}{2} \approx 0.618$ be such that $\phi^2 = 1-\phi$. Define $\gamma := \max\{-\frac{2}{\log\phi}, -\frac{3}{\log\alpha}\} \approx \max\{2.88, 3.94\} = 3.94$, we hence know

$$\begin{aligned} \gamma\log\alpha &\leq -3, & \gamma\log(1-\alpha) &= \gamma\log(\alpha^{5/3}) = \tfrac{5}{3}\gamma\log\alpha \leq -5 \\ \gamma\log\phi &\leq -2, & \gamma\log(1-\phi) &= \gamma\log(\phi^2) = 2\gamma\log\phi \leq -4, \end{aligned}$$

Also set $\delta = 2$. We define three functions that bound the heights that we want to achieve in drawings of various types:

$$h_A(x) = \gamma\log x + \delta \approx 3.94\log(x) + 2, \quad h_B(x) = h_A(x)+2, \quad h_C(x) = h_A(x)+3$$

Finally we denote the height of a drawing Γ by $h(\Gamma)$. Theorem 3 now holds if we show the following result.

Lemma 2 *Let G be a maximal outer-1-planar graph with reference-edge (s,t). Then G has embedding-preserving orthogonal bar-drawings $\mathcal{A}, \overline{\mathcal{B}}, \underline{\mathcal{B}}, \mathcal{C}$ that each have at most two bends per edge and for which the following holds:*

- *$\mathcal{A}$ is of type A and $h(\mathcal{A}) \leq h_A(|G|)$,*
- *$\overline{\mathcal{B}}$ is of type $\overline{B}$ and $h(\overline{\mathcal{B}}) \leq h_B(|G|)$,*
- *$\underline{\mathcal{B}}$ is of type $\underline{B}$ and $h(\underline{\mathcal{B}}) \leq h_B(|G|)$,*
- *$\mathcal{C}$ is of type C and $h(\mathcal{C}) \leq h_C(|G|)$.*

Furthermore, $\min\{h(\overline{\mathcal{B}}), h(\underline{\mathcal{B}})\} \leq h_A(|G|)$.

We prove Lemma 2 by induction on $|G|$ (recall that $|G| = n-1$). In the base case, G consists of only edge (s,t), and one easily constructs suitable drawings. See Figure 4. The height is at most 2 in all cases. Since $|G| = 1$, we have $\log|G| = 0$ and the bounds hold by $\delta = 2$.

4.2 Subgraphs and tools

Now assume that $n \geq 3$, so G has at least one inner cell. The idea is to split G into subgraphs, recursively obtain their drawings, and put them together suitably. The subgraphs will all be "hanging subgraphs", defined as follows. For any uncrossed edge $(u,v) \neq (s,t)$, the *hanging subgraph* H_{uv} is the graph induced by the vertices on the path between v and u on the outer-face, using the path from v to u that does *not* include edge (s,t). This hanging subgraph uses edge (u,v) as its reference-edge.

We have two cases, depending on whether the cell at the reference-edge (s,t) contains a crossing or not; see also Figure 5. In *Case* Δ, the inner cell at (s,t) has no crossing; by maximality it is hence a triangle, say $\{s,t,x\}$. We will recurse on the two hanging subgraphs $H_{s,x}$ and $H_{x,t}$, and use $H_L := H_{s,x}$ and $H_R := H_{x,t}$ as convenient shortcuts. Observe that $|H_L| + |H_R| = |G|$ since we define the size to be one less than the number of vertices. In *Case* $\times$ the inner cell at (s,t) is incident to a crossing, say edge (s,y) crosses edge (t,x). By maximality the edges (s,x), (x,y) and (y,t) exist and have no crossing. We will recurse on the three hanging subgraphs $H_L := H_{s,x}$, $H_M := H_{x,y}$ and $H_R := H_{y,t}$. Observe that $|H_L| + |H_M| + |H_R| = |G|$. These (two or three) subgraphs are smaller, and we assume that they have been drawn recursively, giving drawings $\mathcal{A}_L, \overline{\mathcal{B}}_L, \underline{\mathcal{B}}_L, \mathcal{C}_L$ for subgraph H_L, and similarly for the other subgraphs. In Figures 7-11, we use $\rotatebox[origin=c]{180}{$\mathcal{A}_L$}$ for drawing $\mathcal{A}_L$ rotated by 180 degrees, and similarly for other drawing-types and subgraphs.

To put drawings together, we frequently use two tools from [3, 4]:

- If we have a drawing Γ of some subgraph, then we can insert empty rows to increase the height since the drawing is orthogonal. If we choose the place to add these rows suitably, then this does not change the type of the drawing.
- If we have a drawing Γ of some subgraph, with vertex s in the top row, then we can *release* s: add a new row above Γ, let s occupy all of this row, and re-route the edges at s. Specifically, edges that end vertically at s can simply be extended. At most one edge can end horizontally at s; we re-route its last segment to become vertical (this can only decrease the number of

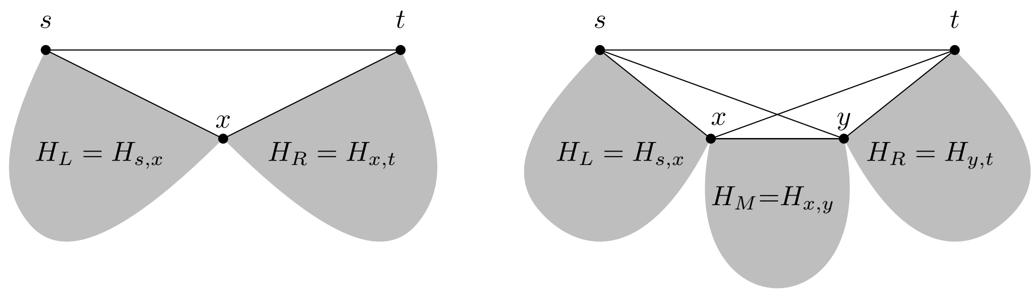


Figure 5: Splitting a subgraph.

bends). See Figure 6. Releasing vertex s increases the height by 1, and achieves that s now occupies both the top left and top right corner in the resulting drawing Γ'.

Similarly we can release vertex t to occupy the bottom-left and bottom-right corner, presuming it was in the bottom row. In the figures, we use a "prime" (e.g. $\mathcal{A}'_L$ as opposed to $\mathcal{A}_L$) to indicate that one pole has been released, and a dotted box for the prior location of this pole.

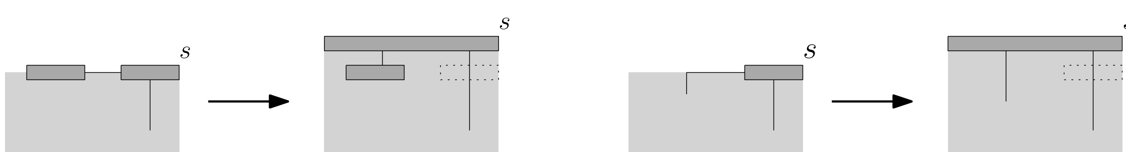


Figure 6: Releasing vertex s.

4.3 Induction step—Case $\times$

We start with Case $\times$ where the cell incident to (s, t) has a crossing, and study the three different types of drawings that we want to achieve.

Case $\times$.A: We want a type-A drawing of height $h_A(|G|)$. We distinguish sub-cases by the size of the hanging subgraph H_M.

Sub-case $\times$.A.1: $|H_M| \leq \alpha|G|$ (recall that $\alpha \approx 0.59$). We know that $|H_L| + |H_R| \leq |G|$, hence we may assume $|H_R| \leq |G|/2$ and use construction $\times.A.(a)$ from Figure 7. (The case $|H_L| \leq |G|/2$ is symmetric and uses construction $\times.A.(b)$.)

We will (for this case only) explain in detail how Figure 7 is to be interpreted; for later cases we hope that the figures alone suffice. We use drawings $\mathcal{A}_L, \mathcal{A}_M$ and $\mathcal{A}_R$ of the subgraphs H_L, H_M, H_R. The primes in the figure indicate that we should release y in both $\mathcal{A}_M$ and $\mathcal{A}_R$ to get $\mathcal{A}'_M$ and $\mathcal{A}'_R$. Rotate $\mathcal{A}'_M$ by 180° to get $\rotatebox{180}{$\mathcal{A}'_M$}$. Increase the height of drawings, if needed, such that $\mathcal{A}'_R$ and $\rotatebox{180}{$\mathcal{A}'_M$}$ have the same height; then combine the two bars of y into one. Increase the height of $\mathcal{A}_L$, if needed, so that it is at least two rows taller than the other two drawings. Then we combine these drawings and route the edges $(s, y), (x, t)$ and (s, t) as shown in Figure 7(a).

One can easily verify that the result is an embedding-preserving drawing with at most two bends per edge. To argue that its height is sufficiently small, the general procedure is as follows.

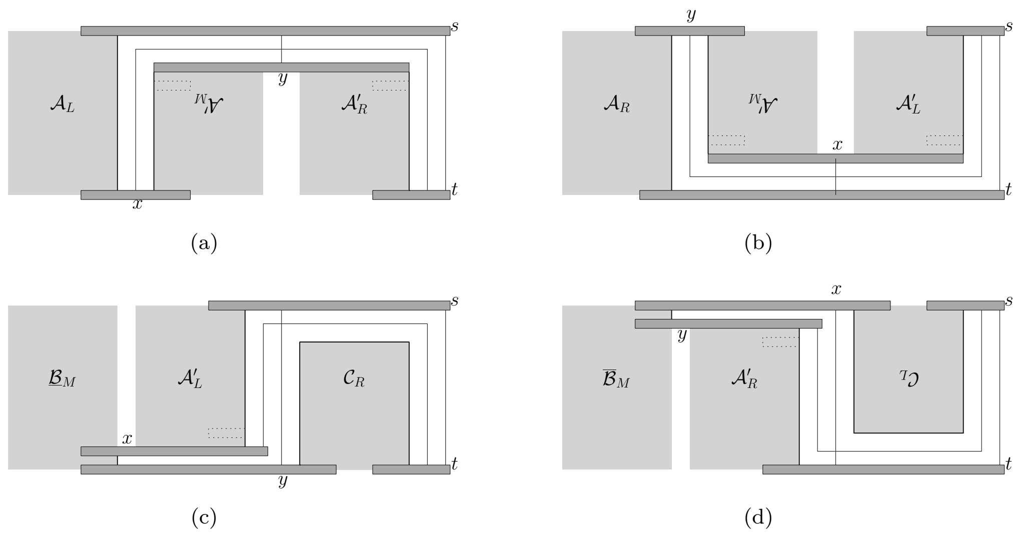


Figure 7: Constructions for $\times.A$.

The drawing is obtained by combining drawings of three (later two) subgraphs. At each subgraph, analyze the *height demand*, i.e., the number of rows needed in the x-range spanned by the subgraph-drawing. (In case $\times.A.1$ we must show that the height demand at each subgraph-drawing is at most $h_A(|G|)$). For this, study the height needed for the subgraph-drawing itself and add more rows (if needed) for releasing vertices and/or routing edges and/or other bars. (For case $\times.A.3$ we add three rows at each of $\mathcal{A}_M$ and $\mathcal{A}_R$ and so need them to have height at most $h_A(|G|) - 3$).

In the specific case here, the height-analysis is done as follows. Since $|H_L| \leq |G|$, we have $h(\mathcal{A}_L) \leq h_A(|H_L|) \leq h_A(|G|)$. We have $|H_M| \leq \alpha|G|$, so

$$\begin{aligned} h(\mathcal{A}_M) &\leq h_A(|H_M|) = \gamma \log |H_M| + \delta \leq \gamma \log(\alpha \cdot |G|) + \delta \\ &= \gamma \log |G| + \delta + \gamma \log \alpha = h_A(|G|) + \gamma \log \alpha \leq h_A(|G|) - 3 \end{aligned}$$

by $\gamma \log \alpha \leq -3$. Since $|H_R| \leq \frac{1}{2}|G| < \alpha|G|$, likewise $\mathcal{A}_R$ has height at most $h_A(|G|) - 3$. We need three more rows above $\mathcal{A}_M$ and $\mathcal{A}_R$: one to release y, one for edge (x, t) and one for s. So the height demand is at most $h_A(|G|)$ at all subgraphs as desired.

Sub-case $\times$.A.2: $|H_M| > \alpha|G|$. We know that one of $\overline{\mathcal{B}}_M$ or $\underline{\mathcal{B}}_M$ has height at most $h_A(|H_M|)$. Let us assume that this is $\underline{\mathcal{B}}_M$, and we then use construction $\times.A.(c)$ from Figure 7 (the other case uses construction $\times.A.(d)$ and is similarly analyzed).

We have $h(\underline{\mathcal{B}}_M) \leq h_A(|H_M|) \leq h_A(|G|)$ by assumption. Since $|H_R| \leq |G| - |H_M| \leq (1-\alpha)|G|$, we have

$$\begin{aligned} h(\mathcal{C}_R) &\leq h_C(|H_R|) \leq \gamma \log((1-\alpha)|G|) + \delta + 3 \\ &= h_A(|G|) + \gamma \log(1-\alpha) + 3 \leq h_A(|G|) - 2 \end{aligned}$$

by $\gamma \log(1-\alpha) \leq -5$. We need two more rows above $\mathcal{C}_R$ (for (x,t) and s). Drawing $\mathcal{A}_L$ has height at most $h_A(|H_L|)$, which by $|H_L| \leq (1-\alpha)|G|$ is similarly shown to be at most $h_A(|G|) - 5$. We need two rows below $\mathcal{A}_L$ (for releasing x and for y). Therefore the height demand is at most $h_A(|G|)$ at all subgraphs.

Case ×.B: We want two drawings, of type $\overline{B}$ and $\underline{B}$. Both must have height at most $h_B(|G|)$, and one must have height at most $h_A(|G|)$.

Consider first constructions $\times.B.(a)$ for a type-$\overline{B}$ drawing, and $\times.B.(b)$ for a type-$\underline{B}$ drawing, see Figure 8. In both, the drawing of H_M has height at most $h_B(|H_M|) \leq h_B(|G|)$. Drawings $\mathcal{A}_L$ and $\mathcal{A}_R$ have height at most $h_A(|G|)$, and we need two more rows at them (one to release a pole, one for a bar of a vertex not in the subgraph). By $h_B(|G|) = h_A(|G|) + 2$ the height demand is at most $h_B(|G|)$ at all subgraphs as desired.

But we must distinguish cases (and perhaps use a different construction) to achieve that one of the drawings has height at most $h_A(|G|)$.

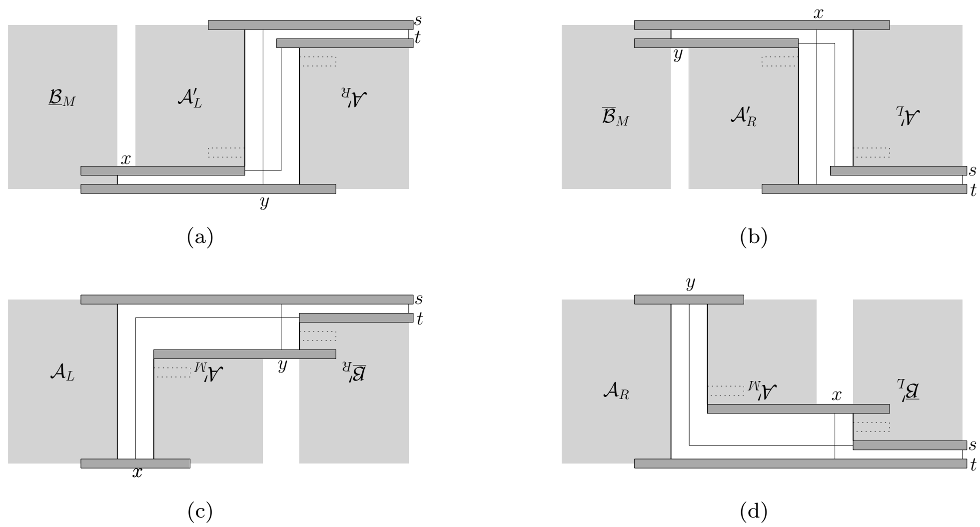


Figure 8: Constructions for $\times.\overline{B}$ and $\times.\underline{B}$.

Sub-case ×.B.1: $|H_L|, |H_R| \leq \phi|G|$ (recall that $\phi = (\sqrt{5}-1)/2 \approx 0.618$). We know that one of $\overline{\mathcal{B}}_M$ or $\underline{\mathcal{B}}_M$ has height at most $h_A(|H_M|)$. Let us assume that this is $\underline{\mathcal{B}}_M$, and consider again construction $\times.B.(a)$ (in the other case one similarly analyzes construction $\times.B.(b)$).

Drawing $\underline{\mathcal{B}}_M$ by assumption has height at most $h_A(|H_M|)$. Also for $i \in \{L, R\}$ we have $|H_i| \leq \phi|G|$ and therefore

$$h(\mathcal{A}_i) \leq h_A(|H_i|) \leq h_A(|G|) + \gamma \log \phi \leq h_A(|G|) - 2$$

since $\gamma \log \phi \leq -2$. We need two further rows at each of $\mathcal{A}_L$ and $\mathcal{A}_R$, so the height demand is at most $h_A(|G|)$ at all subgraphs.

Sub-case ×.B.2: $|H_L| > \phi|G|$. Use construction $\times.B.(c)$ to obtain a type-$\overline{B}$ drawing, see Figure 8. We know that $|H_R| \leq (1-\phi)|G|$ and hence

$$h(\underline{\mathcal{B}}_R) \leq h_B(|H_R|) \leq h_B(|G|) + \gamma \log(1-\phi) \leq h_B(|G|) - 4 = h_A(|G|) - 2$$

since $\gamma \log(1-\phi) \leq -4$. We need two more rows above $\underline{\mathcal{B}}_R$ (one to release t and one for s), so the height demand at $\underline{\mathcal{B}}_R$ is at most $h_A(|G|)$. Similarly the height of $\mathcal{A}_M$ is at most $h(|H_M|) \leq$

$h_A(|G|)-4$. We need four more rows above it: one row for releasing y, one row because $\underline{\mathcal{B}}'_R$ has a row between y and the (released) t, one row for (x,t) and one row for s. So the height demand is at most $h_A(|G|)$ at all subgraphs.

Sub-case $\times$.B.3: $|H_R| > \phi|G|$. Symmetrically construction $\times.B.(d)$ gives a type-$\underline{B}$ drawing of height $h_A(|G|)$.

Case $\times$.C: We want a type-C drawing of height $h_C(|G|)$.

Sub-case $\times$.C.1: $|H_M| \geq \frac{1}{2}|G|$. We know that one of H_L, H_R has size at most $\frac{1}{2}(|G|-|H_M|) \leq \frac{1}{4}|G|$. Let us assume that $|H_L| \leq \frac{1}{4}|G|$, and we then use construction $\times.C.(a)$ from Figure 9 (the other case uses construction $\times.C.(b)$ and is similarly analyzed).

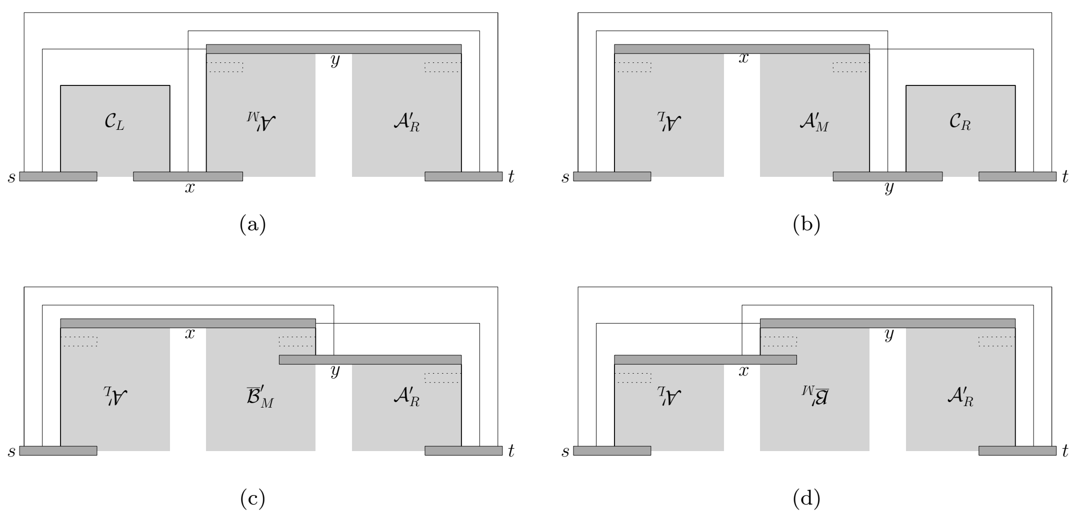


Figure 9: Constructions for $\times.C$.

Drawings $\mathcal{A}_M$ and $\mathcal{A}_R$ both have height at most $h_A(|G|)$, and we need three more rows (one to release y, one for (x,t) and one for (s,t)) so the height demand here is at most $h_A(|G|)+3 = h_C(|G|)$. By $|H_L| \leq \frac{1}{4}|G|$, we have

$$h(\mathcal{C}_L) \leq h_C(|H_L|) \leq h_C(|G|) + \gamma \log(\tfrac{1}{4}) < h_C(|G|) - 4$$

by $\gamma > 2$. We require three more rows above $\mathcal{C}_L$: one for (s,y), one row that was used for (x,t) elsewhere, and one row for (s,t). So the height demand here is less than $h_C(|G|)$.

Sub-case $\times$.C.2: $|H_M| \leq \frac{1}{2}|G|$. We know that one of H_L, H_R has size at most $\frac{1}{2}|G|$. Let us assume that this is H_R, and we then use construction $\times.C.(c)$ from Figure 9 (the other case uses construction $\times.C.(d)$ and is similarly analyzed).

Drawing $\mathcal{A}_L$ has height at most $h_A(|G|)$, and we need three more rows (for releasing x, edge (s,y) and edge (s,t)), so the height demand here is at most $h_A(|G|)+3 = h_C(|G|)$. Also,

$$h(\overline{\mathcal{B}}_M) \leq h_B(|H_M|) \leq h_B(|G|) + \gamma \log(\tfrac{1}{2}) < h_B(|G|) - 2 = h_C(|G|) - 3$$

by $\gamma > 2$. Again we need three more rows, so the height demand is at most $h_C(|G|)$. Similarly by $|H_R| \leq \frac{1}{2}|G|$ we have $h(\mathcal{A}_R) \leq h_A(|H_R|) < h_A(|G|) - 2$. We need five more rows at $\mathcal{A}_R$: one row for releasing y, one row because $\overline{\mathcal{B}}'_M$ had one row between y and the (released) x, one row for (x, t), one row that was used for (s, y) elsewhere, and one row for (s, t). So the height demand is less than $h_A(|G|) + 3 = h_C(|G|)$ here and at most $h_C(|G|)$ at all subgraphs.

4.4 Induction step—Case Δ

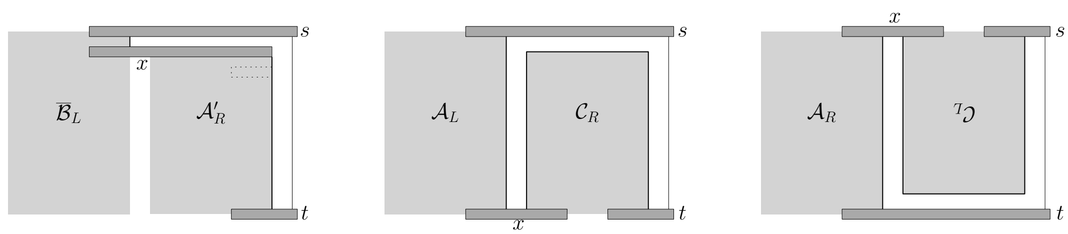
Now we turn our attention to the (much simpler) case Δ where the cell incident to edge (s, t) has no crossing. We again distinguish cases by the drawing-type that we want to achieve.

Case Δ.A: We want a type-A drawing of height $h_A(|G|)$.

Sub-case Δ.A.1: $|H_L|, |H_R| \leq \phi|G|$. Then use construction $\Delta.A.(a)$ from Figure 10. We have

$$h(\overline{\mathcal{B}}_L) \leq h_B(|H_L|) \leq h_B(|G|) + \gamma \log \phi \leq h_B(|G|) - 2 = h_A(|G|)$$

since $\gamma \log \phi \leq -2$. Similarly $h(\mathcal{A}_R) \leq h_A(|H_R|) \leq h_A(|G|) - 2$ and we need two more rows above it (for releasing x and for s). So the height demand is at most $h_A(|G|)$ at all subgraphs.



(a) (b) (c)

Figure 10: Constructions for $\Delta.A$.

Sub-case Δ.A.2: $|H_L| > \phi|G|$. Then use construction $\Delta.A.(b)$. We have $h(\mathcal{A}_L) \leq h_A(|G|)$, and by $|H_R| \leq |G| - |H_L| \leq (1 - \phi)|G|$,

$$h(\mathcal{C}_R) \leq h_C(|H_R|) \leq h_C(|G|) + \gamma \log(1 - \phi) \leq h_C(|G|) - 4 = h_A(|G|) - 1$$

since $\gamma \log(1 - \phi) \leq -4$. We require one more row above $\mathcal{C}_R$ (for s), so the height demand is at most $h_A(|G|)$ at all subgraphs.

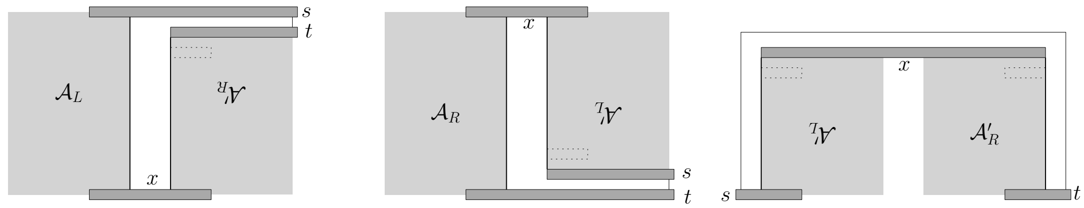
Sub-case Δ.A.3: $|H_R| > \phi|G|$. Symmetrically one proves that construction $\Delta.A.(c)$ has height at most $h_A(|G|)$.

Case Δ.B: We want two drawings, of type $\overline{B}$, $\underline{B}$. Both have height at most $h_B(|G|)$, and one has height at most $h_A(|G|)$.

The construction for the type-$\overline{B}$ drawing is in Figure 11(a). Since $\mathcal{A}_L$ and $\mathcal{A}_R$ have height at most $h_A(|G|)$, and we need two further rows above $\mathcal{A}_R$, the height demand is at most $h_A(|G|) + 2 = h_B(|G|)$ at all subgraphs. If $|H_R| \leq \frac{1}{2}|G|$ then

$$h(\mathcal{A}_R) \leq h_A(|H_R|) \leq h_A(|G|) + \gamma \log(\tfrac{1}{2}) < h_A(|G|) - 2$$

by $\gamma > 2$ and so the height demand is at most $h_A(G)$ at all subgraphs.



(a) (b) (c)

Figure 11: Constructions for $\Delta.\overline{B}$ and $\Delta.\underline{B}$, as well as for $\Delta.C$.

Likewise, the construction of a type-$\underline{B}$ drawing in Figure 11(b) has height at most $h_A(|G|)+2$, and if $|H_L| \leq \frac{1}{2}|G|$ then the height is at most $h_A(|G|)$. Since one of H_L and H_R has size at most $\frac{1}{2}|G|$, one of the drawings has height at most $h_A(|G|)$.

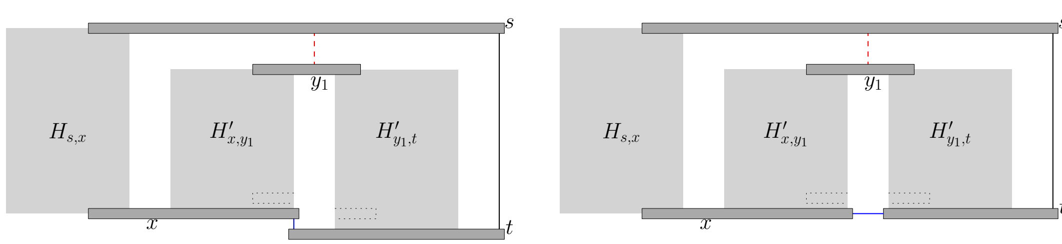
Case $\Delta.C$: We want a type-C drawing of height at most $h_C(|G|)$. The construction is shown in Figure 11(c). Since $\mathcal{A}_L$ and $\mathcal{A}_R$ have height at most $h_A(|G|)$, and we need two more rows above them (to release x and for edge (s,t)), the height is actually at most $h_A(|G|)+2 < h_C(|G|)$.

4.5 Putting it all together

We have given suitable constructions in all cases, so by induction Lemma 2 holds. Using the type-A drawing, we get a drawing of height $3.94\log(n-1)+2$, width $O(n)$, and at most two bends per edge. Therefore Theorem 3 holds. Following the proof, one also sees that the drawing can easily be found in linear time, since we can construct the four drawings of each hanging subgraph in constant time from the drawings of its subgraphs.

4.6 The approach of [2]

We now briefly review the construction by Auer et al. [2] to explain what they overlooked. Their algorithm (just like ours) mimicked the approach of drawing outer-planar graphs from [3, 4], i.e., they fix a reference-edge (s,t), and aim to create a planar visibility representation of type A. To do so, they split the graph into hanging subgraphs depending on the configuration at the cell at (s,t), draw these hanging subgraphs recursively, and merge.



(a) (b)

Figure 12: (a) Putting drawings together in [2]. (d) A variation of (a).

Figure 12(a) shows how Auer et al. [2] put drawings together in case $\times$, and assuming (up to symmetry) that $|H_{s,x}| \geq |H_{x,t}|$. (We use y_1, rather than y, to mirror the notation in [2].) Their construction is somewhat like our case $\times.A.1$ (Figure 7(a)), but they release other poles, mirror drawings rather than rotate them, and therefore can route edge (x, t) without bend. However, there are a few issues with this construction:

- First, to achieve a logarithmic height bound it is crucial that the constructed drawing is no bigger than the drawing of the biggest subgraph. This is violated in Figure 12(a) (where the biggest subgraph is $H_{s,x}$), though the issue can easily be fixed by drawing one edge horizontally instead, see Figure 12(b).

- Second, Auer et al. silently assume that case $\times$ always applies, i.e., the cell incident to (s, t) has a crossing. This issue probably could have been fixed by studying more cases (and/or using more drawing types), since most of our constructions in case Δ do not have bends anyway, and the construction for $\Delta.C$ could be modified to have no bends by releasing other poles and mirroring rather than rotating.

- Finally, even if case $\times$ applies, the construction by Auer et al. does not cover all sub-cases. Specifically, the construction of Figure 12(b) leads to logarithmic height if $H_{s,x}$ contains a constant fraction of the vertices, and a symmetric construction works if $H_{y_1,t}$ contains a constant fraction of the vertices. But Auer et al. did not consider the case where nearly all vertices belong to H_{x,y_1}.

  This issue is the one that led to our counter-example, constructed such that (s, t) is incident to a crossing, subgraph H_{x,y_1} contains all remaining vertices and is built in the same way recursively.

5 Conclusion

In this paper, we pointed out an error in a result by Auer et al., and show that for some outer-1-planar graphs, any poly-line drawing without crossings requires $\Omega(n^2)$ area. We then studied orthogonal box-drawings of outer-1-planar graphs where crossings are permitted. We created such drawings (using bars to represent vertices) that have $O(n \log n)$ area and at most 2 bends per edge, and exactly reflect the given outer-1-planar embedding.

We believe that reducing the number of bends per edge should be possible, and in particular, conjecture that we can achieve $O(n \log n)$ area with at most one bend per edge, perhaps at the expense of modifying the 1-planar embedding. (Very recently, embedding-preserving drawings with one bend per edge and $O(n^{1.48})$ area have been found [6].)

Straight-line drawings of outer-1-planar graphs are also of interest. It is known that there are embedding-preserving outer-1-planar straight-line drawings of area $O(n^2)$ [2]. Are there straight-line drawings of sub-quadratic area (again perhaps at the expense of not respecting the 1-planar embedding)?

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