1 Introduction

A *track layout* of a graph is a partition of its vertices into sequences, called *tracks*, such that the vertices in each sequence form an independent set and no two edges between a pair of tracks cross each other. The *track number* of

Table 1: Track numbers of various families of planar graphs

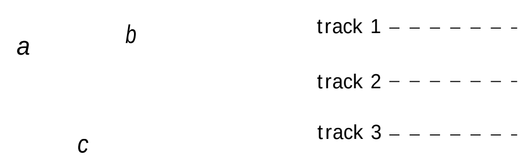
	Upper bound				Lower bound			
Graph class	Old	Ref.	New	Ref.	Old	Ref.	New	Ref.
tree	3	[11]			3	[11]		
outerplanar	5	[9]			4	[9]	**5**	[Thm. 3]
series-parallel	15	[4]			6	[8]		
planar 3-tree	4;000	[1]	**25**	[Thm. 1]	6	[8]	**8**	[Cor. 1]
planar	461;184;080	[7]	**225**	[Thm. 2]	7	[9]	**8**	[Cor. 1]
X-tree	6	[2]	**5**	[Thm. 4]	3	[4]	**5**	[Thm. 5]
Halin	6	[2]			3	[9]	**5**	[Thm. 5]
weakly leveled	6	[2]			3	[2]	**6**	[Thm. 5]

while certain subfamilies (for example, X-trees) admit a layout on 5 tracks. The results close the gaps between upper and lower bounds on the track numbers for the subclasses of planar graphs. Our lower bounds in the section rely on computational experiments using a SAT 87355 Td [-27(erimen)27(ts)-250(u)-416(in)-417(the)-417(section)-416(

(a) A track layout of a tree wrapped onto a 2-clique-colorable 3-track layout. Edges (2-cliques) are partitioned into two nice orders, blue and shaded red.

Consider a parent node, y 2 $V(T)$, of node x in the tree-partition. By Lemma 2, vertices of T_y adjacent to a vertex of T_x form a maximal clique, which we call the *parent clique* of x. The vertices of T_y form a partial k-tree, which by the assumption of the lemma admits a c-clique-colorable t-track layout, $T(T_y)$. Thus, the parent clique of x has an assigned color, $c(x)$, ranging from 1 to c, and cliques with the same color are nicely ordered in $T(T_y)$.

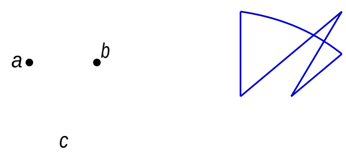
We layout T on (possibly many) tracks such that, for every parent node y



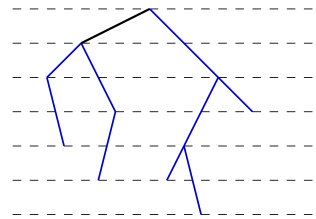
(a) A graph with track number 5 (b) Q-configuration (c) W-configuration

Figure 4: An illustration for Theorem 3

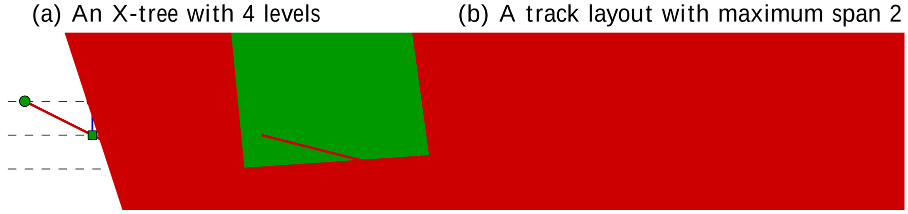
$track(v) = 2$ and $b < v < c$), its neighbor u is not on tracks 1 and 3. We call this


a
b
c

(a) An X-tree with 4 levels



(b) A track layout with maximum span 2



(c) Adding vertices v_{2i-1} and v_{2i} (squares) while maintaining an invariant $y(w_i) \le y(v_{2i-2}) \le y(w_i) + 3$

Figure 6: An illustration for Theorem 4: constructing a 5-track layout for X-trees

Our lower bound examples in the section rely on computational experiments. To this end, we propose a SAT formulation of the track layout problem, and share the source code of our implementation [14]. The formulation is simple-to-implement but efficient enough to find optimal track layouts of graphs with up to a few hundred of vertices in a reasonable amount of time.

5.1 An Upper Bound for X-trees

An *X-tree* is a complete binary tree with extra edges connecting vertices of the same level. Formally, if $v_1, v_2, \ldots, v_{2^d}$

The drawing is built inductively on the level, $d \geq 0$, of a given X-tree using a hypothesis that every X-tree admits a planar straight-line drawing such that:

- $|y(u) - y(v)| = 1$ for every level edge (u, v) and $|y(u) - y(v)| \geq 2$ for every tree edge;

- for the level-d vertices in the tree, $v_1, v_2, \ldots, v_{22}$ 1

6 Conclusions and Open Problems

References