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Simultaneous Embeddings with Few Bends and Crossings*

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Abstract

A *simultaneous embedding with fixed edges* (Sefe) of two planar graphs R and B is a pair of plane drawings of R and B that coincide when restricted to the common vertices and edges of R and B. We show that whenever R and B admit a Sefe, they also admit a Sefe in which every edge is a polygonal curve with few bends and every pair of edges has few crossings. Specifically:

(1) if R and B are trees then one bend per edge and four crossings per edge pair suffice (and one bend per edge is sometimes necessary),

(2) if R is a planar graph and B is a tree then six bends per edge and eight crossings per edge pair suffice, and

(3) if R and B are planar graphs then six bends per edge and sixteen crossings per edge pair suffice.

Our results simultaneously improve on a paper by Grilli et al. (GD'14), which proves that nine bends per edge suffice, and on a paper by Chan et al. (JGAA '15), which proves that twenty-four crossings per edge pair suffice.

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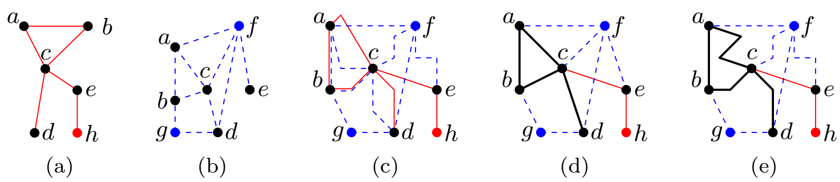


Figure 1: (a-b) R and B with $V_C = \{a, b, c, d, e\}$ and $E_C = \{(a, b), (b, c), (a, c), (c, d)\}$. (c) Simultaneous embedding of R and B. (d) SGE of R and B. (e) SEFE of R and B.

1 Introduction

Let $R = (V_R, E_R)$ and $B = (V_B, E_B)$ be two planar graphs sharing a *common graph* $C = (V_R \cap V_B, E_R \cap E_B)$. The vertices and edges of C are *common*, while the other vertices and edges are *exclusive*. We refer to the edges in $E_R - E_B$, in $E_B - E_R$, and in $E_R \cap E_B$ as the *red*, *blue*, and *black edges*, respectively. A *simultaneous embedding* of R and B is a pair of plane drawings of R and B, respectively, that agree on the common vertices (see Figs. 1a–1c).

Simultaneous graph embeddings have been a central topic of investigation for the graph drawing community in the last decade, because of their applicability to the visualization of dynamic graphs and of multiple graphs on the same vertex set [9, 15], and because of the depth and breadth of the theory they have been found to be related to.

Brass et al. [9] initiated the research on this topic by investigating *simultaneous geometric embeddings* (or SGEs), which are simultaneous embeddings where all edges are represented by straight-line segments (see Figure 1d). This setting proved to be fairly restrictive: there exist two trees [22] and even a tree and a path [4] with no SGE. Furthermore, the problem of deciding whether two graphs admit an SGE is NP-hard [16].

Two relaxations of SGE have been considered in the literature in which edges are not forced to be straight-line segments. In the first setting, we look for a simultaneous embedding of two given planar graphs R and B in which every edge is drawn as a polygonal curve with few bends. Erten and Kobourov [14] proved that three bends per edge always suffice, a bound which has been improved to two bends per edge by Di Giacomo and Liotta [13]. If R and B are trees, then one bend per edge is sufficient [14]. Note that black edges may be represented by different curves in each drawing. The variant in which the edges of R and B might only cross at right angles has also been considered [5]. In the second setting, we look for a *simultaneous embedding with fixed edges* (or SEFE) of R and B: a simultaneous embedding in which every common edge is represented by the same simple curve in the plane (see Figure 1e). In other words, a SEFE is a drawing Γ of the *union graph* $(V_R \cup V_B, E_R \cup E_B)$ such that Γ restricted to the vertices and edges of R is a plane drawing of R and Γ restricted to the vertices and edges of B is a plane drawing of B. While not every two planar graphs admit a SEFE, this setting is substantially less restrictive than SGE: for example, every tree and every planar graph admit a SEFE [18]. Determining the complexity of deciding whether two given graphs admit a SEFE is a major open problem in the field of graph drawing. Polynomial-time testing algorithms are known in many restricted cases, such as when the common graph C is biconnected [3], when C is a set of disjoint cycles [8], or when R is a planar graph and B is a graph with at most one cycle [17]. We refer to an excellent survey by Bläsius et al. [7] for many other results.

In this paper we present algorithms to construct SEFEs in which edges are represented by polygonal curves. For the purpose of guaranteeing the readability of the representation, we aim at minimizing two natural aesthetic measures in the constructed SEFEs: the number of bends per edge and the number of crossings per edge pair. Graph drawings with few bends per edge (see, e.g., [11,21,27,29,30]) and with few crossings (see, e.g., [1,2,6,24,28]) have been studied extensively in the past. Further, both measures have been recently and separately considered in relation to the construction of a SEFE. Namely, Grilli et al. [23] proved that every combinatorial SEFE can be realized as a SEFE with at most nine bends per edge, a bound which improves to three bends per edge when the common graph is biconnected. Further, Chan et al. [10] proved that if R and B admit a SEFE, then they admit a SEFE in which every red-blue edge pair crosses at most twenty-four times.

Contribution. In this paper we improve on the results of Grilli et al. [23] and of Chan et al. [10] by proving the following results.

(1) If R and B are both trees, then they admit a SEFE with one bend per edge. Consequently, every edge pair crosses at most four times. The number of bends is the best possible, since there exist two trees that do no admit a SEFE with no bends [22].

(2) If R is a planar graph and B is a tree, then they admit a SEFE with six bends per edge in which every two exclusive edges cross at most eight times.

(3) If R and B are planar graphs that admit a SEFE, then they admit a SEFE with six bends per edge in which every two exclusive edges cross at most sixteen times.

In all results, the common edges are drawn as straight-line segments. Our algorithms are constructive and can be implemented efficiently in the real-RAM model.

The rest of the paper is organized as follows. In Section 2 we establish some preliminaries. In Sections 3, 4, and 5, we present our results on tree–tree pairs, on tree–planar pairs, and on planar–planar pairs, respectively. Finally, in Section 6 we conclude and suggest some open problems.

2 Preliminaries

A *plane drawing* of a (multi)graph G is a mapping of each vertex to a point in the plane, and of each edge to a simple curve connecting its endvertices such that no two edges cross. A plane drawing of G determines a circular ordering of the edges incident to each vertex of G; the set of these orderings is called a *rotation system*. Two plane drawings of G are *equivalent* if they have the same rotation system, the same containment relationship between cycles, and the same outer face (the second condition is redundant if G is connected). A *planar embedding* is an equivalence class of plane drawings. With a slight abuse of terminology, we say that a planar drawing Γ of a planar graph G is *equivalent* to a planar embedding $\mathcal{E}$ of G, if the Γ is a member of $\mathcal{E}$.

Analogously, a SEFE of two planar graphs R and B determines a circular ordering of the edges incident to each vertex (comprising edges incident to both R and B); the set of these orderings is the *rotation system* of the SEFE. Two SEFEs of R and B are *equivalent* if they have the same rotation system and if their restriction to the vertices and edges of R (of B) determines two equivalent plane drawings of R (of B). Finally, a *combinatorial* SEFE $\mathcal{E}$ for two planar graphs R and B is an equivalence class of SEFEs; we denote by $\mathcal{E}|_R$ (by $\mathcal{E}|_B$) the planar embedding of R (of B) obtained by restricting $\mathcal{E}$ to the vertices and edges of R (of B). Again, with a slight abuse of terminology, we say that a SEFE Γ of two planar graphs R and B is *equivalent* to a combinatorial SEFE $\mathcal{E}$ of R and B if Γ is a member of $\mathcal{E}$.

A *subdivision* of a multigraph G is a graph G' obtained by replacing edges of G with paths, whose internal vertices are called *subdivision vertices*. If G' is a subdivision of G, the operation of *flattening* subdivision vertices in G' returns G. The *contraction* of an edge (u, v) in a multigraph G leads to a multigraph G' by replacing (u, v) with a vertex w incident to all the edges u and v are incident to in G; k parallel edges (u, v) in G lead to $k-1$ self-loops incident to w in G' (the contracted edge itself is not in G'). If G has a planar embedding $\mathcal{E}_G$, then G' *inherits* a planar embedding $\mathcal{E}_{G'}$ as follows. Let $a_1, \dots, a_k, v$ and $b_1, \dots, b_\ell, u$ be the clockwise orders of the neighbors of u and v in $\mathcal{E}_G$, respectively. Then the clockwise order of the neighbors of w is $a_1, \dots, a_k, b_1, \dots, b_\ell$. The contraction of a connected graph is the contraction of all its edges.

The straight-line segment between points p and q is denoted by $\overline{pq}$. The *angle* of $\overline{pq}$ is the angle between the ray from p in positive x-direction and the ray from p through $\overline{pq}$. A polygon P is *strictly-convex* if, for any two non-consecutive vertices p and q of P, the open segment $\overline{pq}$ lies in the interior of P; also, P is *star-shaped* if a point p^* exists such that, for any vertex p of P, the open segment $\overline{pp^*}$ lies in P; the *kernel* of P is the set of all such points p^*.

Consider a straight-line drawing Γ of a graph G and a subgraph T of G which is a tree. Suppose that the drawing of T in Γ is planar. A *surrounding curve* γ for T in Γ is a closed curve drawn on top of Γ that contains T inside and that is "close enough" to the drawing of T in Γ so that: (i) no vertex other than those of T lies inside γ; (ii) no crossing of Γ lies inside γ; and (iii) every edge e that is incident to a vertex of T and that is not an edge of T crosses γ a

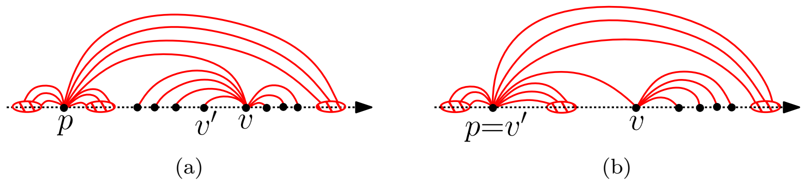


Figure 2: Embedding the children of v if (a) $v' \neq p$ or (b) $v' = p$. Parts of the embedding already constructed are in the dashed regions.

number of times equal to the number of endvertices of e in T.

A *1-page book embedding* (1PBE) is a plane drawing where all vertices are placed on an oriented line ℓ called *spine* and all edges are curves in the halfplane to the left of ℓ. A *2-page book embedding* (2PBE) is a plane drawing where all vertices are placed on ℓ and each edge is a curve in one of the two halfplanes delimited by ℓ.

The following lemma shows how to construct a constrained 1PBE of a tree.

Lemma 1 *Let T be a tree with a planar embedding $\mathcal{E}$. For every vertex v of T, let e_v be a designated edge incident to v. There is a 1PBE for T equivalent to $\mathcal{E}$ such that for every vertex v of T, the spine passes through v right before e_v in clockwise order around v.*

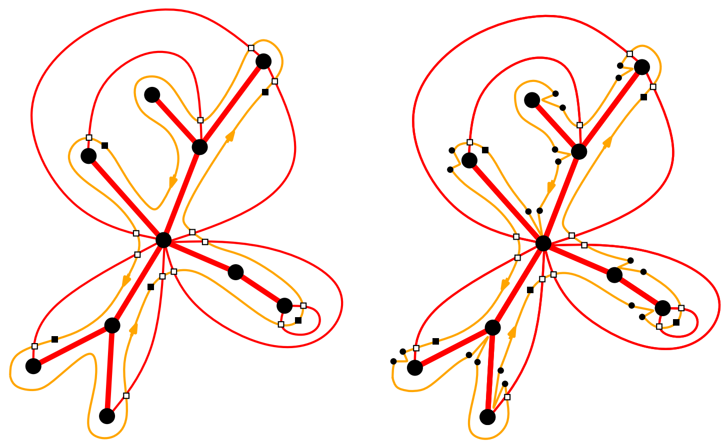
Proof: We construct the embedding recursively. Arbitrarily choose a vertex s as the root of T and place s on the spine ℓ. Place the other endvertex of e_s after s on ℓ and all remaining neighbors of s, if any, in between in the order given by $\mathcal{E}$. Then process every child v of s (and the subtree below v) recursively as follows (and ensure that all subtrees stay in pairwise disjoint parts of the spine, for instance, by assigning a specific region to each).

Note that both v and the parent p of v are already embedded. Assume that p lies before v on the spine, the case in which p lies after v is analogous. Let v' be the endvertex of e_v different from v. If $v' \neq p$, then we place it right before v. Both if $v' \neq p$ (see Figure 2a) and if $v' = p$ (see Figure 2b), we place the other children of v, if any, according to $\mathcal{E}$, in the parts of the spine between p and v', and after v. If v is not a leaf, then all its children are processed recursively in the same fashion. It is easily checked that the resulting embedding is a 1PBE that satisfies the stated properties. □

The following lemma shows how to make a planar multigraph Hamiltonian by subdividing each edge a few times.

Lemma 2 *Let G be a connected planar multigraph with a planar embedding $\mathcal{E}$. For every vertex v of G, let e_v be a designated edge incident to v. There exists a simple planar graph G' such that:*

- Two-subdivision*: G' is obtained by subdividing each edge of G either zero or two times and by adding dummy vertices and edges to the resulting graph;*



(a) (b)

Figure 3: Illustration for Lemma 2. (a) The graph G with its planar embedding $\mathcal{E}$ has black disks as vertices and red curves as edges. The curve γ is orange. Subdivision vertices for the edges of G are white squares and dummy vertices are black squares. (b) The graph G' with its planar embedding $\mathcal{E}'$. Both the red and the orange curves represent edges of G'.

- EMBEDDING*: G' has a planar embedding $\mathcal{E}'$ from which $\mathcal{E}$ is obtained by removing dummy vertices and edges and flattening subdivision vertices;*
- HAMILTONIAN CYCLE*: G' contains a Hamiltonian cycle $\mathcal{C}$, which we orient counter-clockwise in $\mathcal{E}'$, none of whose edges is (part of) an edge of G;*
- STARTING EDGE*: for every vertex v of G, the edge of $\mathcal{C}$ entering v comes immediately before e_v in the clockwise order of edges incident to v in $\mathcal{E}'$; and*
- START INSIDE*: all the edges of G' that are incident to a vertex of G and that are part of an edge of G lie inside $\mathcal{C}$ in $\mathcal{E}'$.*

Proof: Let T be a spanning tree of G, which exists since G is connected. Let γ be a surrounding curve for T in $\mathcal{E}$. Insert subdivision vertices for the edges of G not in T at these crossings; also, for every two subdivision vertices that are consecutive along γ and that subdivide the same edge of G, insert a dummy vertex on γ between them. Orient γ counter-clockwise. See Figure 3.a.

We now modify γ in a small neighborhood of each vertex v of G, so that it passes through v. If e_v is in T, as in Figure 4.a, then while traversing γ counter-clockwise stop at a point in which γ follows e_v towards v; insert a dummy vertex at that point, then let γ take a detour from the dummy vertex to v and then back to its previous route, where another dummy vertex is inserted. If

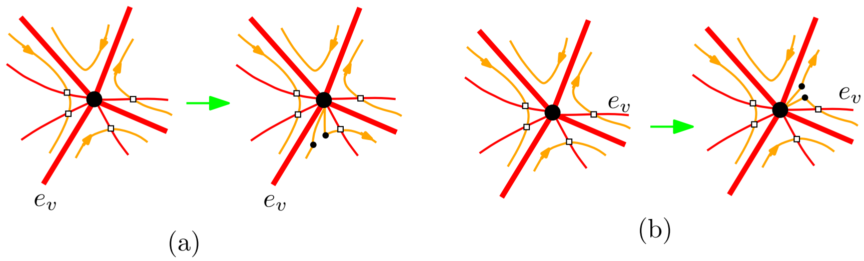


Figure 4: Illustration for Lemma 2. Modifying γ so that it passes through a vertex v.

e_v is not in T, as in Figure 4.b, then while traversing γ counter-clockwise stop immediately after the crossing between γ and e_v that is "closer" to v; insert a dummy vertex on γ at that point, then let γ take a detour from the dummy vertex to v and then back to its previous route, where another dummy vertex is inserted. Finally, we consider γ as a cycle, that is, each curve that is part of γ and that connects two consecutive vertices on γ is an edge. Denote by G' the resulting graph and by $\mathcal{E}'$ its planar embedding; see Figure 3.b.

It is easy to verify that G' and $\mathcal{E}'$ satisfy all the required properties. In particular, the Hamiltonian cycle $\mathcal{C}$ required by the statement is the cycle corresponding to γ: By construction, $\mathcal{C}$ passes through every vertex of G and it does so immediately before e_v in clockwise order around v. Also, every edge of G has been subdivided twice (if it is not in T) or never (if it is in T). Further, all the edges of T lie inside γ and hence inside $\mathcal{C}$, while all the edges of G not in T start inside $\mathcal{C}$, move outside it, and then end again inside it; this implies Properties Hamiltonian cycle and Start inside. Finally, G' is simple, due to the introduction of dummy vertices along γ. $\square$

3 Two Trees

In this section we describe an algorithm that computes a Sefe of any two trees R and B with one bend per edge. Let C be the common graph of R and B.

The outline of the algorithm is as follows. In Step 1, we compute a combinatorial Sefe of R and B with the property that at every common vertex v, all the black edges are consecutive in the circular order of edges incident to v. In Step 2, we contract each component of C to a single vertex, obtaining trees R' from R and B' from B. In Step 3, we independently augment R' and B' to Hamiltonian planar graphs, so as to satisfy topological constraints that are necessary for the subsequent drawing algorithms. In Step 4, we use the Hamiltonian augmentations to construct a simultaneous embedding of R' and B' with one bend per edge; this step is reminiscent of an algorithm of Erten and Kobourov [14]. Finally, in Step 5, we expand the components of C. This consists of modifying the simultaneous embedding of R' and B' in a small neighborhood

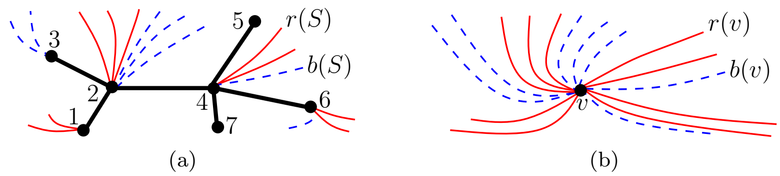


Figure 5: (a) A connected component S of C, together with its incident exclusive edges. (b) Vertex v resulting from the contraction of S.

of each vertex to make room for the components of C. We now describe these steps in detail.

Step 1: Combinatorial Sefe. Fix the clockwise order of the edges incident to each vertex as follows: all the black edges in any order, then all the red edges in any order, and then all the blue edges in any order (each sequence might be empty). As any rotation system for a tree determines a planar embedding for it, this results in a combinatorial SEFE $\mathcal{E}$ of R and B. See Figure 5a. We may assume that every component S of C is incident to at least one red and one blue edge: If S is not incident to any, say, blue edge, then $B = S = C$, since B is connected; hence, any plane straight-line drawing of R is a SEFE of R and B.

For every component S of C we pick two incident edges $r(S)$ and $b(S)$ as follows. Consider any SEFE $\Gamma_{\mathcal{E}}$ of R and B equivalent to $\mathcal{E}$; let γ be a surrounding curve for S in $\mathcal{E}$. Note that γ intersects all the exclusive edges incident to S in some clockwise order; further, in this order all the exclusive edges incident to a single vertex of S appear consecutively. Let $r(S)$ be any red edge not preceded by a red edge in this order and let $b(S)$ be the first blue edge after $r(S)$. We define a total ordering ϱ_S of the vertices of S, as the order in which their exclusive edges intersect γ (a curve is added incident to every vertex of S with no incident exclusive edge for this purpose), where the first vertex of ϱ_S is the endvertex of $r(S)$. We have the following.

Lemma 3 *The straight-line drawing of S obtained by placing its vertices on a strictly-convex curve λ in the order defined by ϱ_S is plane.*

Proof: Consider the plane drawing of S in $\Gamma_{\mathcal{E}}$. For every vertex v of S, shrink γ along an exclusive edge incident to v so that γ passes through v and still every edge of S lies in its interior. Eventually γ passes through all the vertices of S in the order ϱ_S. The planarity of the drawing of S in $\Gamma_{\mathcal{E}}$ implies that there are no two edges (u_1, u_2) and (w_1, w_2) whose endvertices appear along γ in the circular order u_1, w_1, u_2, and w_2. Then placing the vertices of S on λ in the order ϱ_S leads to a plane straight-line drawing of S. □

Step 2: Contractions. Contract each component S of C to a single vertex v. The resulting trees $R'=(V'_R, E'_R)$ and $B'=(V'_B, E'_B)$ have planar embeddings

$\mathcal{E}_{R'}$ and $\mathcal{E}_{B'}$ inherited from $\mathcal{E}_R$ and $\mathcal{E}_B$, respectively. The vertex v is common to R' and B'; let $r(v)$ and $b(v)$ be the edges corresponding to $r(S)$ and $b(S)$ after the contraction. See Figure 5b.

Step 3: Hamiltonian augmentations. We describe this step for R' only; the treatment of B' is analogous and independent. The goal is to find a vertex order corresponding to a 1PBE of R'. All edges between consecutive vertices along the spine ℓ, as well as the edge between the first and last vertex along ℓ, can be added to a 1PBE while maintaining planarity, hence the 1PBE determines a Hamiltonian planar augmentation of R'. We need such an augmentation to respect the planar embedding $\mathcal{E}_{R'}$ of R' and, for every common vertex v, to place $r(v)$ right after the spine in clockwise order around v. This is possible due to Lemma 1. Indeed, by means of Lemma 1 we construct a 1PBE for R' equivalent to $\mathcal{E}_{R'}$ such that for every common vertex v, the spine passes through v right before $r(v)$ in clockwise order around v. Note that, in order to apply Lemma 1, we also need to designate, for each exclusive vertex v, an edge incident to v; this can be done arbitrarily.

Step 4: Simultaneous embedding. We now construct a simultaneous embedding of R' and B'. In such an embedding let σ_v denote the linear order of the edges around each vertex v obtained by sweeping a ray clockwise around v, starting in direction of the negative x-axis. Our algorithm is very similar to algorithms due to Brass et al. [9] and Erten and Kobourov [14]. These algorithms, however, do not guarantee the construction of a simultaneous embedding in which the order of the edges incident to each vertex is as stated in the following lemma. This order is essential for the upcoming expansion step.

Lemma 4 *For every $\varepsilon > 0$, the trees R' and B' admit a simultaneous embedding with one bend per edge in which:*

- *all edges of $E_{R'}$ ($E_{B'}$) incident to a vertex v in V'_R (V'_B) exit v within an angle of $[-\varepsilon, +\varepsilon]$ with respect to the positive y-direction (x-direction);*
- *the drawing restricted to R' (to B') is equivalent to $\mathcal{E}_{R'}$ (to $\mathcal{E}_{B'}$); and*
- *for every common vertex v, the first red (blue) edge in σ_v is $r(v)$ ($b(v)$).*

Proof: Refer to Fig. 6. We assign the x-coordinates $1, \dots, |V_{R'}|$ (and the y-coordinates $|V_{B'}|, \dots, 1$) to the vertices of R' (of B') according to the order in which they occur on the spine in the 1PBE of R' (of B') computed in Lemma 1. This determines the placement of every vertex in $V_{R'} \cap V_{B'}$. Set any not-yet-assigned coordinate to 0.

We draw the edges of R' as follows; the construction for B' is analogous. The idea is to realize the 1PBE of R' with its vertices placed as above and its edges drawn as x-monotone polygonal curves with one bend. We proceed as follows. The 1PBE of R' defines a partial order of the edges corresponding to the way they nest. For example, denoting the vertices by their order along the spine,

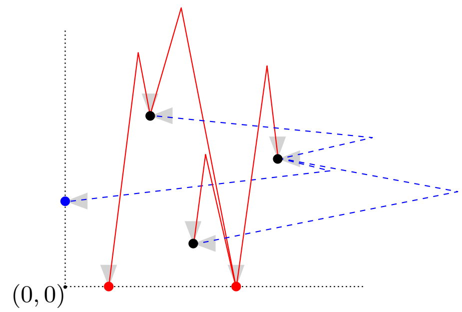


Figure 6: A simultaneous embedding of the trees R' and B'. The gray cones indicate the angles within $[-\varepsilon, +\varepsilon]$ of the positive x- and y-directions.

edge $(3,4)$ precedes $(3,5)$ and $(2,5)$, while $(1,2)$ and $(6,7)$ are incomparable. We draw the edges of R' in any linear extension of this partial order. Suppose we have drawn some edges and let (u,v) be the next edge to be drawn. Assume w.l.o.g. that the x-coordinate of u is smaller than the one of v. For some $\varepsilon_{uv} > 0$, consider the ray ϱ_u emanating from u with an angle of $\pi/2 - \varepsilon_{uv}$ (with respect to the positive x-axis). Similarly, let ϱ_v be the ray emanating from v with an angle of $\pi/2 + \varepsilon_{uv}$. We choose $\varepsilon_{uv} < \varepsilon$ sufficiently small so that:

(1) no vertex in $V_{R'} \setminus \{u\}$ lies in the region to the left of the underlying (oriented) line of ϱ_u and to the right of the vertical line through u;

(2) no vertex in $V_{R'} \setminus \{v\}$ lies in the region to the right of the underlying (oriented) line of ϱ_v and to the left of the vertical line through v; and

(3) neither ϱ_u nor ϱ_v intersects any previously drawn edge.

Since no two vertices of R' have the same x-coordinate and since the edges are drawn in the described order, we can choose ε_{uv} as claimed. The corresponding rays ϱ_u and ϱ_v intersect in some point: this is where we place the bend-point of (u,v). The resulting drawing is equivalent to the 1PBE of R' and therefore to $\mathcal{E}_{R'}$. The remaining claimed properties are preserved from the 1PBE. $\square$

Step 5: Expansion. We now expand the components of C in the drawing produced by Lemma 4 one by one in any order. Let Γ be the current drawing, v be a vertex corresponding to a not-yet-expanded component S of C, and p be the point on which v is placed in Γ. Note that the red and blue edges incident to v may be incident to different vertices in S. Let $\sigma_v = (e_1, \ldots, e_\ell)$, where $e_1, \ldots, e_k$ are red and $e_{k+1}, \ldots, e_\ell$ are blue. By Lemma 4, $r(v) = e_1$ and $b(v) = e_{k+1}$. Each edge incident to v is drawn as a polygonal curve with one bend. Let b_i be the bend-point of e_i. The plan is to delete the segments $\overline{pb_i}$ in Γ to obtain a drawing Γ'. Then, for some small value of ε, draw S in Γ' inside a small disk D_ε with radius ε centered at p and draw segments from S

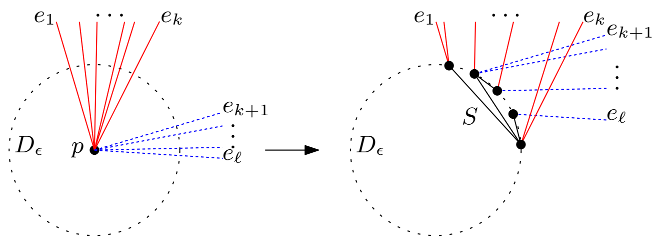


Figure 7: Expanding a component S in a small disk D_ε around p.

to $b_1, \ldots, b_\ell$. See Figure 7. Let Γ_R (Γ'_R) be the restriction of Γ (Γ') to the red and black edges.

We first deal with possible crossings involving a segment from S to $b_1, \ldots, b_\ell$ and a segment of Γ'_R not incident to any vertex of S (see Proposition 1 below); we then deal with possible crossings involving two segments from S to $b_1, \ldots, b_\ell$ (see Proposition 2 below). We state Propositions 1 and 2 only for the red graph; the propositions for the blue graph are analogous. By continuity, v can be moved around slightly in Γ_R while maintaining a plane drawing for the red graph. This implies the following.

Proposition 1 *There exists a $\delta_R > 0$ with the following property. For every δ such that $0 < \delta \leq \delta_R$ and for every drawing Γ^*_R obtained from Γ'_R by drawing S in D_δ, the red segments from S to $b_1, \ldots, b_k$ do not cross any segment already present in Γ'_R.*

The following proposition deals with crossings between red edges incident to S.

Proposition 2 *There exists an $\varepsilon_R > 0$ with the following property. For every ε such that $0 < \varepsilon \leq \varepsilon_R$ and for every k (not necessarily distinct) points $q_1, \ldots, q_k$ in clockwise order on the upper semicircle of D_ε, the segments $\overline{q_1 b_1}, \ldots, \overline{q_k b_k}$ do not intersect each other, except at common endpoints.*

Proof: The angles of $\overline{pb_1}, \ldots, \overline{pb_k}$ are distinct and strictly decreasing, by Lemma 4 and by the way $e_1, \ldots, e_k$ are labeled. We claim that ε_R can be chosen sufficiently small so that the angles of $\overline{q_1 b_1}, \ldots, \overline{q_k b_k}$ are also distinct and strictly decreasing. For a certain ε, let $I_i(\varepsilon)$ be the interval of all angles α such that the ray with angle α from b_i intersects D_ε. Since the angles of $\overline{pb_1}, \ldots, \overline{pb_k}$ are distinct, it follows that the intervals $I_1(0), \ldots, I_k(0)$ are disjoint. By continuity, there exists an $\varepsilon_R > 0$ for which $I_1(\varepsilon_R), \ldots, I_k(\varepsilon_R)$ are also disjoint, and the claim follows for this ε_R. Finally, two segments $\overline{q_i b_i}$ and $\overline{q_j b_j}$ with $i < j$ and $q_i \neq q_j$ can intersect only if the angle of $\overline{q_i b_i}$ is smaller than the angle of $\overline{q_j b_j}$, which does not happen by the claim. □

We get the following main lemma.

Lemma 5 *There exists an $\varepsilon > 0$ with the following property. We can expand S to obtain a simultaneous embedding Γ^* from Γ' by drawing the vertices of S on* *the boundary of D_ε, the edges of S as straight-line segments, and by connecting S to $b_1, \ldots, b_\ell$ with straight-line segments.*

Proof: Let δ_R, δ_B, ε_R, and ε_B be the constants satisfying Propositions 1 and 2 and their analogous formulations for B. Let $\varepsilon := \min\{\delta_R, \delta_B, \varepsilon_R, \varepsilon_B\}$. Place the vertices of S as distinct points on the boundary of the upper-right quadrant of D_ε in the total order ϱ_S of the vertices of S defined earlier. By Lemma 3, this placement determines a straight-line plane drawing of S. Draw straight-line segments from the vertices of S to $b_1, \ldots, b_\ell$, thus completing the drawing of the exclusive edges incident to S. We prove that the red segments incident to S do not cross any red or black edge; the proof for the blue segments is analogous. By Proposition 1, the red segments incident to S do not cross the red and black segments not incident to vertices in S. Also, they do not cross the edges of S, which are internal to D_ε. Further, Proposition 2 ensures that these segments do not cross each other. Namely, the linear order of the vertices of S defined by the sequence of the red edges $e_1, \ldots, e_k$ is a subsequence of ϱ_S, given that the embedding $\mathcal{E}_{R'}$ of R' is the one inherited from $\mathcal{E}_R$, given that Lemma 4 produces a drawing of R' respecting $\mathcal{E}_{R'}$ and in which $e_1 = r(v)$, and given that the endvertex of $r(S)$ in S is the first vertex of ϱ_S. $\square$

We are now ready to state the main theorem of this section.

Theorem 1 *Let R and B be two trees. There exists a* SEFE *of R and B in which every exclusive edge is a polygonal curve with one bend, every common edge is a straight-line segment, and every two exclusive edges cross at most four times.*

Proof: By Lemma 4, the trees R' and B' admit a simultaneous embedding with one bend per edge. By repeated applications of Lemma 5, the simultaneous embedding of R' and B' can be turned into a SEFE of R and B in which every exclusive edge has one bend and every common edge is a straight-line segment. Finally, any two exclusive edges cross at most four times, given that each of them consists of two straight-line segments. $\square$

4 A Planar Graph and a Tree

In this section we give an algorithm which computes a SEFE of any planar graph $R = (V_R, E_R)$ and any tree $B = (V_B, E_B)$ in which every edge of R has at most six bends and every edge of B has one bend.

Without loss of generality, we can assume that $V_R = V_B$ and that R and B are connected. These two conditions can be guaranteed as follows. First, add to V_R (to V_B) the vertices in $V_B - V_R$ (in $V_R - V_B$) as isolated vertices. Second, if R is not connected, then pick a vertex $v \in V_R$ and add to E_R edges from v to a vertex of each connected component of R not containing v; now R is connected. Finally, perform a similar augmentation for B. The common graph $C = (V_R = V_B, E_R \cap E_B)$ of R and B is a forest, as it is a subgraph of B.

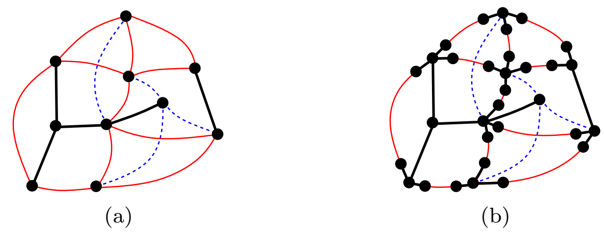


Figure 8: (a) A planar graph R and a tree B. (b) The planar graph R' and the tree B' obtained after Step 1.

Our algorithm has strong similarities with the one for trees (Section 3); however, it encounters some of the complications one needs to handle when dealing with pairs of general planar graphs (Section 5). Its outline is as follows.

In Step 1 we modify R and B to obtain a planar graph R' and a tree B' with a common graph C' (whose vertex set coincides with the ones of R' and B') by introducing *antennas*, which are edges of C' replacing parts of the exclusive edges of R. While this costs two extra bends per edge of R in the final SEFE of R and B, it establishes the property that, for every exclusive edge e of R', every endvertex of e is incident to two edges only, namely e and an edge of C'.

In Step 2 we construct a combinatorial SEFE of the planar graph R' and the tree B' such that, at every vertex v, all the edges of C' are consecutive in the circular order of the edges incident to v. The existence of such a combinatorial SEFE is guaranteed by the fact that the exclusive edges of R' satisfy the property mentioned above.

In order to construct a SEFE of R' and B', in Steps 3–6 we perform a *(a) contraction (b) simultaneous embedding (c) expansion* process similar to the one in Section 3. This again relies on an independent Hamiltonian augmentation of the graphs, which is done in Step 4. The augmentation of the tree B' is done by Lemma 1; in order to augment the planar graph R' we need to subdivide some of its edges; this results in the use of four bends per edge in the SEFE of R' and B'. Finally, we obtain a SEFE of R and B by removing the antennas. Next we describe these steps in detail.

Step 1: Antennas. We replace every exclusive edge $e = (u, v) \in E_R$ by a path (u, u_e, v_e, v), with two new common vertices u_e and v_e, black edges $(u, u_e), (v_e, v)$, and a red edge (u_e, v_e). See Figure 8. The resulting planar graph R' and tree B' satisfy the following property.

Property 1 *For every exclusive edge e of R', every endvertex of e is incident to two edges only, namely e and an edge of the common graph C'.*

We also get the following:

Lemma 6 *Suppose that a* SEFE Γ' *of* R' *and* B' *exists in which: (i) every edge of* R' *(of* B'*) is a polygonal curve with at most* x *bends (*y *bends); (ii) every common edge is a straight-line segment; and (iii) any two exclusive edges cross at most* z *times.*

Then there exists a SEFE Γ *of* R *and* B *in which: (i) every edge of* R *(of* B*) is a polygonal curve with at most* $x+2$ *bends (*y *bends); (ii) every common edge is a straight-line segment; and (iii) any two exclusive edges cross at most* z *times.*

Proof: We obtain Γ from Γ' by removing all the edges (u, u_e) and (v_e, v) from the drawing of the tree B, and by interpreting all the vertices u_e and v_e as bend-points in the drawing of R.

First, we have that Γ is a SEFE of the planar graph R and of the tree B. In particular, every two edges of B are also edges of B' and since they do not cross in Γ', they do not cross in Γ either. Further, each edge of R corresponds to a path in R', hence no two edges of R cross in Γ as the corresponding paths do not cross in Γ'.

Second, every edge of C is also an edge of C', hence it is a straight-line segment in Γ, as it is in Γ'.

Third, every edge of B is also an edge of B', hence it is a polygonal curve with at most y bends in Γ, as it is in Γ'.

Fourth, every edge e in R corresponds to a path in R' composed of at most two edges of C', which are straight-line segments, and of one exclusive edge of R', which has at most x bends. Hence, e has at most $x+2$ bends in Γ (the two extra bends correspond to the points where u_e and v_e are).

Finally, any exclusive edge of R or B corresponds to at most two edges of C' and of one exclusive edge of R' or B'. Since common edges are crossing-free, any two exclusive edges of R and B cross the same number of times as the corresponding exclusive edges of R' and B', which is z by assumption. $\square$

Step 2: Combinatorial Sefe. Start with any plane drawing of the planar graph R'. This determines the planar embeddings $\mathcal{E}_{R'}$ of R' and $\mathcal{E}_{C'}$ of C'. The planar embedding $\mathcal{E}_{B'}$ of the tree B' is defined by setting the clockwise order of the edges incident to each vertex v in $\mathcal{E}_{B'}$ to be: all the black edges incident to v first (in the order defined by $\mathcal{E}_{R'}$) and then all the blue edges incident to v in any order. In the resulting combinatorial SEFE $\mathcal{E}$ of R' and B' we have, in clockwise order around each vertex, either: (i) a sequence of black edges followed by a sequence of blue edges (where one of the sequences might be empty); or (ii) a single black edge followed by a single red edge. This is a consequence of Property 1 and of the embedding choices for B'. As in Section 3, we can assume that every connected component of C' is incident to at least one red and one blue edge. We choose the edges $r(S)$ and $b(S)$ for every component S of C' and we define an ordering ϱ_S of the vertices of S as in Section 3.

Step 3: Contractions. Contract each component of C' to a vertex in R' and in B', determining graphs $R'' = (V_{R''}, E_{R''})$ and $B'' = (V_{B''}, E_{B''})$, respectively, where $V_{R''} = V_{B''}$. Note that R'' is a planar multigraph, i.e., it might have parallel edges and self-loops, while B'' is a tree. In this way R'' inherits a planar embedding $\mathcal{E}_{R''}$ from $\mathcal{E}_{R'}$ and B'' inherits a planar embedding $\mathcal{E}_{B''}$ from $\mathcal{E}_{B'}$. For each vertex v of R'' and B'', let $r(v)$ and $b(v)$ be the edges corresponding to $r(S)$ and $b(S)$ after the contraction.

Step 4: Hamiltonian augmentations. A Hamiltonian planar augmentation of the tree B'' is constructed as described in the proof of Lemma 1 (where, for each vertex v, the edge $b(v)$ plays the role of the designated edge e_v).

A Hamiltonian planar augmentation of the planar multigraph R'' might not exist, hence before performing any augmentation we need to subdivide some edges of R''. This is done by means of Lemma 2, applied with $G = R''$, $e_v = r(v)$, and $\mathcal{E} = \mathcal{E}_{R''}$. The lemma allows us the construction of a (simple) Hamiltonian planar graph $R''' = G'$ and of a planar embedding $\mathcal{E}_{R'''} = \mathcal{E}'$ satisfying Properties Two-subdivision, Embedding, Hamiltonian cycle, Starting edge, and Start inside. Denote by $V_{R'''}$ and $E_{R'''}$ the vertex and edge set of R''', respectively.

Step 5: Simultaneous embedding. Ideally, in order to construct a simultaneous embedding of R'' and B'', we would like to use known algorithms that construct simultaneous embeddings with two bends per edge of every two planar graphs [12, 13, 26]. However, the existence of self-loops in R'' prevents us from doing that. In the following lemma we show how to modify those algorithms to deal with non-simple graphs. Figure 9 shows an example of the resulting drawing. Let σ_v be defined as in Section 3.

Lemma 7 *For every $\varepsilon > 0$, the graph R'' and the tree B'' admit a simultaneous embedding in which:*

- *every edge of R'' (of B'') is a polygonal curve with at most four bends (with one bend);*
- *every two edges cross at most eight times (counting an adjacency as one crossing);*
- *all edges of $E_{R''}$ ($E_{B''}$) incident to a vertex v in $V_{R''}$ ($V_{B''}$) exit v within an angle of $[-\varepsilon, +\varepsilon]$ with respect to the positive y-direction (x-direction);*
- *the drawing restricted to R'' (to B'') is equivalent to $\mathcal{E}_{R''}$ (to $\mathcal{E}_{B''}$); and*
- *for every vertex v, the first red (blue) edge in σ_v is $r(v)$ ($b(v)$).*

Proof: We first construct a simultaneous embedding of R'' and B''; we will later prove that it satisfies the required properties.

We start by placing the vertices of R''' and B'' in the plane. Similarly to Lemma 4, we assign x-coordinates $1, \dots, |V_{R'''}|$ to the vertices of R''' according to the order in which they occur along the Hamiltonian cycle of R''', starting at any vertex u^*. Further, we assign y-coordinates $|V_{B''}|, \dots, 1$ to the vertices of

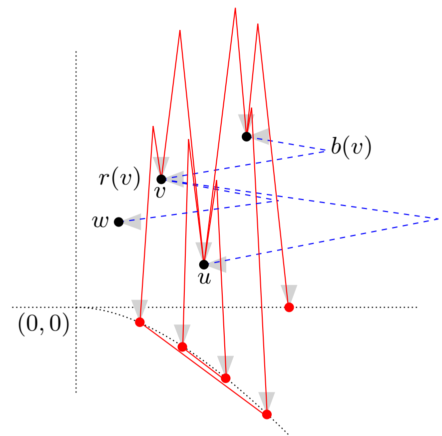


Figure 9: A simultaneous embedding of a planar graph R'' and a tree B''. The gray cones indicate the angles within $[-\varepsilon, +\varepsilon]$ of the positive x- and y-directions. The self-loop at u in R'', represented as a path of length three in R''', crosses the edge (v, w) eight times. Some angles in the drawing were modified slightly to reduce the height of the figure.

B'' according to the order in which they occur on the spine in the Hamiltonian augmentation of B'' that is computed by Lemma 1. This determines the placement of every vertex in $V_{R''} = V_{B''}$. It remains to assign y-coordinates to the vertices in $V_{R'''} \setminus V_{R''}$ (note that none of these vertices belongs to B''). A subset V_s of the vertices in $V_{R'''} \setminus V_{R''}$ consists of subdivision vertices for the edges in $E_{R''}$; we assign y-coordinates to the vertices in V_s so that they lie on the curve $y = -x^2$. We set the y-coordinate of every vertex in $V_{R'''} \setminus \{V_{R''} \cup V_s\}$ to 0.

We continue by drawing the edges of R'' and B''. The edges of B'' are drawn exactly as in Lemma 4. We draw the edges of R'' as follows. Note that the Hamiltonian augmentation of R'' corresponds to a 2PBE of R''' along a spine ℓ, where u^* can be assumed w.l.o.g. to be the first vertex along ℓ. This 2PBE defines a partition of the edges of R''' into those embedded in the half-plane $\mathcal{H}_l$ to the left of ℓ and those embedded in the half-plane $\mathcal{H}_r$ to the right of ℓ. By Property START INSIDE, each edge of R''' in $\mathcal{H}_r$ connects two vertices in V_s, which are subdivision vertices for an edge in $E_{R''}$; thus the edges of R''' in $\mathcal{H}_r$ form a perfect matching on V_s. We draw these edges as straight-line segments. In order to draw the edges of R''' in $\mathcal{H}_l$, we define a partial order on these edges, corresponding to the way they nest. We draw these edges one by one, in any linear extension of this partial order. The procedure to draw an edge (u, v) as a 1-bend edge is the same as in Lemma 4. That is, assuming w.l.o.g. that u has a smaller x-coordinate than v, the bend-point is the intersection point between two rays ϱ_u and ϱ_v emanating from u and v with an angle of $\pi/2 - \varepsilon_{uv}$ and $\pi/2 + \varepsilon_{uv}$, for some suitably small $0 < \varepsilon_{uv} < \varepsilon$. The vertices in $V_{R'''} \setminus \{V_{R''} \cup V_s\}$ are removed from the drawing, together with their incident edges, while the vertices in V_s are interpreted as bend-points. This determines a drawing $\Gamma_{R''}$ of R''.

We prove that $\Gamma_{R''}$ and the constructed drawing $\Gamma_{B''}$ of B'' constitute a simultaneous embedding of R'' and B'' as required by the lemma. First, observe that $\Gamma_{B''}$ satisfies all the required properties, as in Lemma 4. We now deal with $\Gamma_{R''}$.

- *Angle at v*: All the edges of R'' incident to a vertex v in $V_{R''}$ exit v within an angle of $[-\varepsilon, +\varepsilon]$ with respect to the positive y-direction; this is true by Property Start inside of Lemma 2, by the fact that edges of R''' inside the Hamiltonian cycle of R''' are in $\mathcal{H}_l$ in the 2PBE, and by the construction just described for the edges of R''' in $\mathcal{H}_l$.

- *Equivalence to $\mathcal{E}_{R''}$*: $\Gamma_{R''}$ is equivalent to $\mathcal{E}_{R''}$ by Property Embedding of Lemma 2 and since $\mathcal{E}_{R'''}$ determines the 2PBE which the construction of the drawing of R'' relies upon.

- *First edge in σ_v*: For every common vertex v, the first red edge, if any, in σ_v is $r(v)$, by Property Starting edge in Lemma 2.

- *Number of bends*: Each edge of R'' either coincides with an edge of R''' or consists of three edges of R''', depending on whether it is subdivided zero or two times in the proof of Lemma 2. If an edge of R'' coincides with an edge of R''', then it has one bend in $\Gamma_{R''}$. If it is composed of three edges of R''', then it has four bends in $\Gamma_{R''}$, namely one, zero, and one bend on the three edges of R''' composing it and lying in $\mathcal{H}_l$, $\mathcal{H}_r$, and $\mathcal{H}_l$ in the 2PBE, respectively, plus two bends corresponding to its subdivision vertices.

- *Planarity*: The vertices in V_s are placed along the curve $y = -x^2$ in $\Gamma_{R''}$, in the same order as they occur along ℓ. Hence, the edges lying in $\mathcal{H}_r$ in the 2PBE do not cross each other in $\Gamma_{R''}$. That no two edges lying in $\mathcal{H}_l$ in the 2PBE cross each other in $\Gamma_{R''}$ can be argued as in Lemma 4. Finally, any edge lying in $\mathcal{H}_l$ in the 2PBE has no intersection with the interior of the convex hull of the vertices in V_s (provided that ε is small enough). Hence, it has no intersection in $\Gamma_{R''}$ with any edge lying in $\mathcal{H}_r$ in the 2PBE.

It remains to bound the number of crossings between any edges e_r of R'' and e_b of B''. Note that e_b is composed of two straight-line segments in $\Gamma_{B''}$. If e_r is also composed of two straight-line segments, then e_r and e_b cross at most four times. Otherwise, e_r is composed of five straight-line segments, from which an upper bound of ten on the number of crossings between e_r and e_b directly follows. This bound is improved to eight by observing that the third segment of e_r (corresponding to the edge of R''' lying in $\mathcal{H}_r$) does not cross the two segments composing e_b, as the former lies in the open half-plane $y < 0$, while the latter lie in the closed half-plane $y \geq 0$. □

Step 6: Expansion. Next, we expand the components of C' in the simultaneous embedding of R'' and B'' obtained by Lemma 7. This expansion is performed by means of Lemma 5, exactly as in Section 3. That is, the components of C' are expanded one by one; when a component S is expanded, its vertices are placed in the order ϱ_S on the upper-right quadrant of the boundary of a suitably small disk D_ε centered at the vertex S was contracted to. This results in a SEFE of R' and B'. Finally, the vertices and edges not in R and B are removed, in order to get a SEFE of R and B.

Note that the expansion described above is only possible because the edges of C' incident to each vertex v are consecutive in circular order around v in the planar embeddings $\mathcal{E}_{R'}$ of R' and $\mathcal{E}_{B'}$ of B'. While this property could be guaranteed even for the tree B, the planar graph R might not admit any planar embedding satisfying it. This is the reason for the introduction of the antennas, which guarantee Property 1 and hence that the edges of C' incident to v are consecutive in circular order around v in any planar embedding $\mathcal{E}_{R'}$ of R'.

Theorem 2 *Let R be a planar graph and let B be a tree. There exists a* SEFE *of R and B in which every exclusive edge of R is a polygonal curve with at most six bends, every exclusive edge of B is a polygonal curve with one bend, every common edge is a straight-line segment, and every two exclusive edges cross at most eight times.*

Proof: By Lemma 7, R'' and B'' admit a simultaneous embedding Γ'' in which every edge of R'' (of B'') is a polygonal curve with at most four bends (with one bend). By repeated application of Lemma 5, the simultaneous embedding Γ'' can be turned into a SEFE Γ' of R' and B' in which every exclusive edge of R' (of B') has at most four bends (one bend) and every common edge is a straight-line segment. By Lemma 6, there exists a SEFE Γ of R and B in which every exclusive edge of R (of B) has at most six bends (one bend) and every common edge is a straight-line segment. Concerning the number of crossings, by Lemma 7 every two edges cross at most eight times in Γ'', also counting their adjacencies. While the expansions performed in Lemma 5 in order to construct Γ' starting from Γ'' might introduce new proper crossings for a pair of exclusive edges of R' and B', they only do so at the cost of removing the adjacency between the corresponding edges of R'' and B''. Hence, the maximum number of crossings per pair of edges is eight in Γ' and, by Lemma 6, is eight also in Γ. □

5 Two Planar Graphs

In this section we give an algorithm to compute a SEFE of any two planar graphs R and B in which every edge has at most six bends, provided that a combinatorial SEFE $\mathcal{E}$ of R and B exists and is given to us as part of the input. This assumption is necessary, since testing the existence of a SEFE of two planar graphs is a problem of unknown complexity [7].

Let C be the common graph of R and B. We assume that no exclusive vertex or edge lies in the outer face of C in $\mathcal{E}$, and that R and B are connected. These two conditions are indeed met after the following augmentation. First, introduce a 3-cycle δ^* in C (whose vertices and edges did not belong to R and B before) and embed it in $\mathcal{E}$ so that it surrounds the rest of R and B. Then, introduce a red (blue) vertex inside each face f of R (of B) in $\mathcal{E}$ different from the outer face, and connect it to all the vertices incident to f.

We outline our algorithm. First, R and B are modified into planar graphs R' and B' with a common graph C' by introducing antennas, as in Step 1 of the algorithm in Section 4; however, here the modification is performed for both graphs. Similarly to Sections 3 and 4, we would like to *contract* each component S of C' to a vertex, construct a *simultaneous embedding* of the resulting graphs, and finally *expand* the components of C'. However, S is here not just a tree, but rather a planar graph containing in its internal faces other components of C' (and exclusive vertices and edges of R' and B'). Hence, the *(a) contraction (b) simultaneous embedding (c) expansion* process does not happen just once, but rather we proceed from the outside to the inside of C' iteratively, each time applying that process to draw certain subgraphs of R' and B', until R' and B' have been entirely drawn. We now describe this algorithm in more detail.

First, we introduce antennas in R and B, that is, we replace each exclusive edge $e = (u, v)$ of R (of B) with u and v in C by a path (u, u_e, v_e, v) such that u_e, v_e, (u, u_e), and (v_e, v) are in C, while (u_e, v_e) is exclusive to R (to B). We also replace each exclusive edge $e = (u, v)$ of R (of B) with u in C and v not in C by a path (u, u_e, v) such that u_e and (u, u_e) are in C, while (u_e, v) is exclusive to R (to B). The resulting planar graphs R' and B' satisfy the following property.

Property 2 *For every exclusive edge e, every endvertex of e in the common graph C' of R' and B' is incident to two edges only, namely e and an edge of C'.*

We also get the following lemma, whose proof is analogous to the one of Lemma 6 and hence is omitted here.

Lemma 8 *Suppose that a* Sefe *Γ' of R' and B' exists in which: (i) every edge of R' (of B') is a polygonal curve with at most x bends (y bends); (ii) every common edge is a straight-line segment; and (iii) any two exclusive edges cross at most z times.*

Then there exists a Sefe *Γ of R and B in which: (i) every edge of R (of B) is a polygonal curve with at most $x+2$ bends ($y+2$ bends); (ii) every common edge is a straight-line segment; and (iii) any two exclusive edges cross at most z times.*

Note that a Sefe of R' and B' is naturally derived from a Sefe of R and B by drawing the antennas as "very small" curves on top of the edges they partially replace. Applying such a modification to any Sefe of R and B equivalent to $\mathcal{E}$ results in a Sefe of R' and B' equivalent to a combinatorial Sefe $\mathcal{E}'$ of R' and B'. By Property 2, in $\mathcal{E}'$ we have, in clockwise order around each vertex of

C', either: (i) a sequence of black edges; or (ii) a single black edge followed by a single red edge; or (iii) a single black edge followed by a single blue edge. Let $\mathcal{E}_{C'}$ be the restriction of $\mathcal{E}'$ to C'.

We now iteratively construct a SEFE of R' and B'. We start by representing the cycle δ^* of C' as a strictly-convex polygon Δ^*. Next, assume that a SEFE Γ'' of two subgraphs R'' of R' and B'' of B' with a common graph C'' has been constructed; initially, this assumption is verified with $\Gamma'' = \Delta^*$ and $R'' = B'' = C'' = \delta^*$. Let $\mathcal{E}_{R''}$, $\mathcal{E}_{B''}$, and $\mathcal{E}_{C''}$ be the planar embeddings of R'', B'', and C'' in $\mathcal{E}'$, respectively. We assume that Γ'' satisfies the following invariants.

- BENDS AND CROSSINGS: every edge of R'' or B'' is a polygonal curve with at most four bends, every edge of C'' is a straight-line segment, and every exclusive edge of R'' crosses every exclusive edge of B'' at most sixteen times;

- EMBEDDING: the restrictions of Γ'' to the vertices and edges of R'', B'', and C'' are equivalent to $\mathcal{E}_{R''}$, $\mathcal{E}_{B''}$, and $\mathcal{E}_{C''}$, respectively; and

- POLYGONS: every simple cycle δ_f of C'' that delimits a face f of $\mathcal{E}_{C''}$ from the outside and that contains vertices of R' or B' in its interior in $\mathcal{E}'$ is represented in Γ'' by a star-shaped empty polygon Δ_f; further, if an edge exists in C' that lies inside δ_f in $\mathcal{E}'$ and that belongs to the same 2-connected component of C' as δ_f, then Δ_f is a strictly-convex polygon.

These invariants are indeed satisfied when $R'' = B'' = C'' = \delta^*$ and $\Gamma'' = \Delta^*$. In particular, all the vertices and edges of R' and B' that are not part of δ^* lie inside δ^* in $\mathcal{E}'$, because of the initial augmentation; further, the interior of Δ^* in Γ'' is empty. It remains to describe how to insert in Γ'' some vertices and edges of R' and B' that are not yet in Γ'', while maintaining the above invariants. Since R' and B' are finite graphs, this will eventually lead to a SEFE of R' and B'.

In the following we distinguish two cases. In Case 1 we plug into Γ'' a drawing of a 2-connected subgraph S_f of C' such that a cycle δ_f of S_f is already drawn in Γ''. In Case 2 such a subgraph S_f does not exist and then we plug into the drawing Γ'' a drawing of all the vertices and edges of C' incident to a face f of $\mathcal{E}_{C''}$, as well as a drawing of all the vertices and edges of R' and B' that lie inside f in $\mathcal{E}'$. We now provide details for these two cases.

Case 1: There exists a simple cycle δ_f in C'' that delimits a face f of $\mathcal{E}_{C''}$ from the outside and there exists an edge e_f in C' that lies inside δ_f in $\mathcal{E}'$ and that belongs to the same 2-connected component of C' as δ_f.

By Property POLYGONS, δ_f is represented by a strictly-convex empty polygon Δ_f in Γ'', as in Figure 10a. Consider the maximal 2-connected subgraph S_f of C' whose outer face in $\mathcal{E}'$ is delimited by δ_f; note that e_f is an edge of S_f. As observed in [25], a straight-line plane drawing Γ_f of S_f exists in which the outer face of S_f is delimited by Δ_f and every internal face is delimited by a star-shaped polygon. Plug Γ_f in Γ'', so that they coincide along Δ_f, obtaining

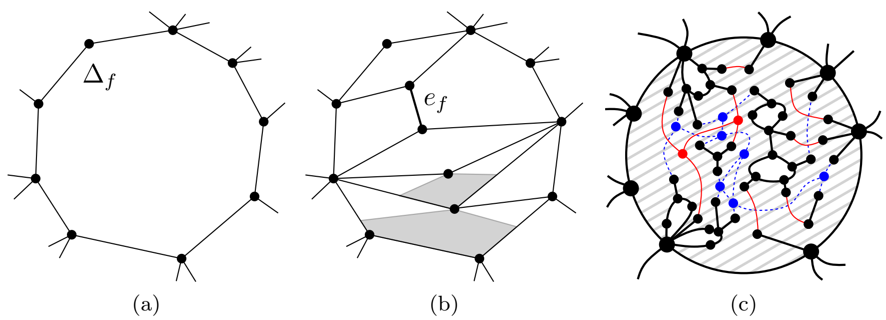


Figure 10: (a) The strictly-convex polygon Δ_f representing δ_f in Γ''. (b) Plugging Γ_f into Γ''. The kernels of the star-shaped (non-convex) polygons delimiting internal faces of S_f in Γ_f are gray. (c) Graphs C'_f, R'_f, and B'_f; δ_f is drawn by thick lines and f is dashed.

a drawing Γ''', as in Figure 10b. Let C''' be the subgraph of C' that is drawn in Γ''' (that is, C''' consists of C'' and of S_f, which share δ_f); further, let $\mathcal{E}_{C'''}$ be the planar embedding of C''' in $\mathcal{E}'$.

Properties Bends and crossings and Embedding are clearly satisfied by Γ'''. Concerning Property Polygons, it suffices to consider any simple cycle $\delta_{f,i}$ of C''' that delimits a face of $\mathcal{E}_{C'''}$ from the outside, that lies inside δ_f in $\mathcal{E}'$, and that contains vertices of R' or B' in its interior in $\mathcal{E}'$. The polygon $\Delta_{f,i}$ representing $\delta_{f,i}$ in Γ''' is star-shaped, by construction. Further, $\Delta_{f,i}$ is empty in Γ''', because Δ_f is empty in Γ''. Finally, no edge exists in C' lying inside $\delta_{f,i}$ in $\mathcal{E}'$ and belonging to the same 2-connected component of C' as $\delta_{f,i}$, as any such an edge would belong to S_f.

Case 2: If Case 1 does not apply, then every simple cycle δ_f in C'' that delimits a face of $\mathcal{E}_{C''}$ from the outside also delimits a face f of $\mathcal{E}_{C'}$ from the outside; other vertices and edges of C' might be incident to f, although no path exists in C' that connects two vertices of δ_f and that lies inside δ_f. By Property Polygons, assuming that Γ'' is not yet a Sefe of R' and B', there exists a simple cycle δ_f in C'' that delimits a face f of $\mathcal{E}_{C''}$ from the outside and that contains vertices of R' or B' in its interior in $\mathcal{E}'$; further, δ_f is represented in Γ'' by a star-shaped empty polygon Δ_f, as in Figure 10c. Let $C'(f)$ be the subgraph of C' composed of the vertices and edges incident to f in $\mathcal{E}_{C'}$; note that $C'(f)$ is a *cactus graph*, which is a graph whose vertices and edges are all incident to a common face, in this case f. Also, let $R'(f)$ be the subgraph of R' composed of $C'(f)$ and of the red vertices and edges lying in f in $\mathcal{E}'$; graph $B'(f)$ is defined analogously. Let $\mathcal{E}_{R'(f)}$, $\mathcal{E}_{B'(f)}$, and $\mathcal{E}_{C'(f)}$ be the restrictions of $\mathcal{E}'$ to $R'(f)$, $B'(f)$, and $C'(f)$, respectively. We have the following main lemma:

Lemma 9 *There exists a* sefe Γ'_f *of* $R'(f)$ *and* $B'(f)$ *with the following properties:*

- *every edge is a polygonal curve with at most four bends, every common edge is a straight-line segment, and every two exclusive edges cross at most sixteen times;*
- Γ'_f *restricted to* $R'(f)$, $B'(f)$, *and* $C'(f)$ *is equivalent to* $\mathcal{E}_{R'(f)}$, $\mathcal{E}_{B'(f)}$, *and* $\mathcal{E}_{C'(f)}$, *respectively;*
- *the cycle* δ_f *is represented by* Δ_f*; and*
- *every simple cycle of* $C'(f)$ *different from* δ_f *is represented by a strictly-convex empty polygon in* Γ'_f.

In order to prove Lemma 9, we present an algorithm consisting of four steps, that resemble Steps 3–6 of the algorithm in Section 4. Note that $R'(f)$ and $B'(f)$ are both connected, since R' and B' are connected. We can hence assume that every component S of $C'(f)$ is incident to at least one red and one blue edge, we can choose edges $r(S)$ and $b(S)$, and we can define an ordering ϱ_S of the vertices of S as in Section 3.

Step 1: Contraction. Contract each component S of $C'(f)$ to a single vertex v. The resulting planar multigraphs $R''(f)$ and $B''(f)$ have planar embeddings $\mathcal{E}_{R''(f)}$ and $\mathcal{E}_{B''(f)}$ inherited from $\mathcal{E}_{R'(f)}$ and $\mathcal{E}_{B'(f)}$. Vertex v is common to $R''(f)$ and $B''(f)$. Let $r(v)$ and $b(v)$ be the edges corresponding to $r(S)$ and $b(S)$ after the contraction. We stress the fact that a component S^* of $C'(f)$ contains cycle δ_f. While S^* is contracted to a vertex u^* as every other component of $C'(f)$, it will later play a special role. We also remark that, unlike the other components, the order of the edges incident to S^* is reversed after the contraction. That is, consider $\mathcal{E}_{R'(f)}$ and $\mathcal{E}_{B'(f)}$, draw a simple closed curve γ in the interior of f arbitrarily close to S^*, and consider the *counter-clockwise* order in which the edges of $R'(f)$ and $B'(f)$ intersect γ; then this is also the *clockwise* order in which the same edges are incident to u^* in $\mathcal{E}_{R''(f)}$ and $\mathcal{E}_{B''(f)}$.

Step 2: Hamiltonian augmentations. We compute Hamiltonian augmentations $R'''(f)$ of $R''(f)$ and $B'''(f)$ of $B''(f)$. This is done independently for $R''(f)$ and $B''(f)$ by means of Lemma 2, applied first with $G = R''(f)$, $e_v = r(v)$, and $\mathcal{E} = \mathcal{E}_{R''(f)}$, and then with $G = B''(f)$, $e_v = b(v)$, and $\mathcal{E} = \mathcal{E}_{B''(f)}$.

Step 3: Simultaneous embedding. A simultaneous embedding of $R''(f)$ and $B''(f)$ is constructed by means of an algorithm very similar to the one in the proof of Lemma 7. Let σ_v be defined as in Section 3. We have the following.

Lemma 10 *For every* $\varepsilon > 0$, *the graphs* $R''(f)$ *and* $B''(f)$ *admit a simultaneous embedding* Γ''_f *in which:*

- *every edge of* $R''(f)$ *and* $B''(f)$ *is a polygonal curve with at most four bends;*

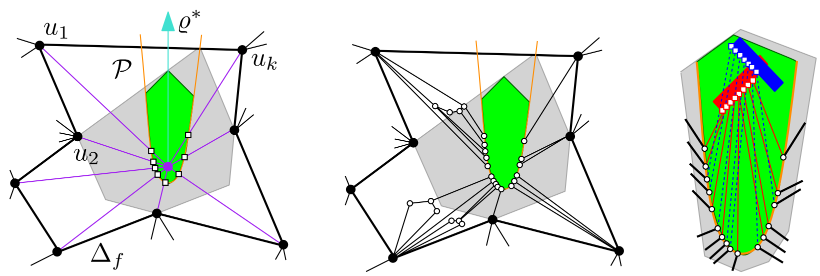
- *every two edges cross at most sixteen times (counting an adjacency as one crossing);*
- *all edges of $R''(f)$ ($B''(f)$) incident to a vertex v in $R''(f)$ ($B''(f)$) exit v within an angle of $[-\varepsilon, +\varepsilon]$ with respect to the positive y-direction (x-direction);*
- *Γ''_f restricted to $R''(f)$ ($B''(f)$) is equivalent to $\mathcal{E}_{R''(f)}$ ($\mathcal{E}_{B''(f)}$);*
- *for every common vertex v of $R''(f)$ and $B''(f)$, the first red (blue) edge in σ_v is $r(v)$ ($b(v)$); and*
- *the vertex u^* is at the point $(1, |V(B'''(f))|)$; the straight-line segments incident to u^* in the drawing of $R''(f)$ (of $B''(f)$) have their endpoints different from u^* on the straight line $x = 1.5$ ($y = |V(B'''(f))| - 0.5$); every other vertex or bend of an edge of $R''(f)$ (of $B''(f)$) is to the right of (below) that line.*

Proof: The algorithm to draw $R''(f)$ and $B''(f)$ is similar to one presented in the proof of Lemma 7 to draw R''. Assign the vertices of $R'''(f)$ (of $B'''(f)$) with distinct x-coordinates $1, \dots, R'''(f)$ (with distinct y-coordinates $1, \dots, B'''(f)$) according to their order in the Hamiltonian cycle of $R'''(f)$ (according to the reverse order in the Hamiltonian cycle of $B'''(f)$). It is important here that u^* is the vertex of $R'''(f)$ (of $B'''(f)$) that gets the smallest x-coordinate (the largest y-coordinate). Place the subdivision vertices for the edges of $R''(f)$ (of $B''(f)$) on the curve $y = -x^2$ ($x = -y^2$); finally, set any non-assigned coordinate to 0. The edges of $R'''(f)$ and $B'''(f)$ are drawn as the edges of R''' in the proof of Lemma 7, except for the edges incident to u^*. Namely, the bend-point of an edge (u^*, v) of $R'''(f)$ (of $B'''(f)$) is placed at the intersection point between the line $x = 1.5$ ($y = |V(B'''(f))| - 0.5$) and the ray ϱ_v emanating from v with an angle of $\pi/2 + \varepsilon_{u^*v}$ (ε_{u^*v}), for some suitably small $0 < \varepsilon_{u^*v} < \varepsilon$.

Remove from the drawing the vertices of $R'''(f)$ and $B'''(f)$ that are neither vertices of $R''(f)$ or $B''(f)$, nor subdivision vertices for the edges of $R''(f)$ or $B''(f)$, and interpret the subdivision vertices for the edges of $R''(f)$ and $B''(f)$ as bend-points. This results in a SEFE Γ''_f of $R''(f)$ and $B''(f)$, which can be proved to satisfy the required properties exactly as in the proof of Lemma 7. In particular, if edges e_r of $R''(f)$ and e_b of $B''(f)$ have four bends each, then they cross at most twenty-five times. This bound can be improved to sixteen, since the third segment of e_r does not cross any of the five segments composing e_b (given that the former lies in the open half-plane $y < 0$, while the latter lie in the closed half-plane $y \geq 0$), and vice versa. □

Step 4: Expansion. This step is more involved than in Sections 3 and 4, because when expanding the component S^*, we need to ensure that the cycle δ_f is drawn as Δ_f.

We first expand the components $S \neq S^*$ of $C'(f)$ in Γ''_f. Differently from the previous sections, S is a cactus graph, rather than a tree. However, Lemma 3



(a) (b) (c)

Figure 11: (a) Geometry inside Δ_f: p^* is purple, $\mathcal{H}^*$ is green, the kernel of Δ_f is gray, and points $p_1, \ldots, p_k$ are empty squares. (b) Drawing Γ^*; vertices of S^* not in δ_f are white disks. (c) Reconnecting Γ''_f to Γ^*; drawing Γ''_f is represented by a red and a blue rectangle. Red and blue squares represent bendpoints of $R'''(f)$ and $B'''(f)$ adjacent to u^*.

holds true (with the same proof) even if S is a cactus graph. Hence, we expand the components $S \neq S^*$ one by one in Γ''_f. When a component S is expanded, its vertices are placed in the order ϱ_S on the upper-right quadrant of the boundary of a suitably small disk D_ε centered at the vertex S was contracted to. Denote again by Γ''_f the resulting SEFE in which every component $S \neq S^*$ of $C'(f)$ has been expanded. Note that every simple cycle of each component $S \neq S^*$ is a strictly-convex empty polygon in Γ''_f, since its incident vertices lie on a strictly-convex curve, namely the boundary of D_ε.

In order to complete the construction of a SEFE Γ'_f of $R'(f)$ and $B'(f)$ as requested by Lemma 9, it remains to deal with the cactus graph S^* containing δ_f. We sketch the plan: we define an open region $\mathcal{H}^*$ inside Δ_f (Figure 11a); we construct a drawing Γ^* of S^* such that δ_f is represented as Δ_f and all the other vertices and edges of S^* are inside Δ_f but outside or on the boundary of $\mathcal{H}^*$ (Figure 11b); we rotate and scale Γ''_f and place it in $\mathcal{H}^*$; finally, we connect Γ^* with Γ''_f via straight-line segments, thus obtaining Γ'_f (Figure 11c).

We begin by defining $\mathcal{H}^*$. Denote by $u_1, \ldots, u_k$ the counter-clockwise order of the vertices along δ_f. Removing the edges of δ_f disconnects S^* into k cactus graphs; w.l.o.g. assume that u_1 is in the one of these cactus graphs that is incident to $r(S^*)$. By Property POLYGONS, Δ_f is star-shaped, hence it has a non-empty kernel. Let p^* be any point in this kernel and ϱ^* be a ray emanating from p^* through u_1. Rotate ϱ^* clockwise around p^* by a sufficiently small angle so that no vertex of Δ_f is encountered during the rotation. For sake of simplicity of description, assume that the origin of the Cartesian axes is at p^*, with ϱ^* being the positive y-axis. Draw a parabola $\mathcal{P}$ with equation $y = ax^2 - b$, with $a, b > 0$; a is large enough and b is small enough so that $\mathcal{P}$ intersects all of $\overline{p^*u_1}, \ldots, \overline{p^*u_k}$ at points $p_1, \ldots, p_k$, and so that a wedge $\mathcal{W}^*$ with angle $\pi/2$, centered at a point on ϱ^*, and bisected by a ray in the negative y-direction exists containing all of $p_1, \ldots, p_k$ in its interior and having an intersection $\mathcal{H}^*$

with the region $y > ax^2 - b$ entirely lying in the kernel of Δ_f. Let $\mathcal{P}^*$ denote the part of $\mathcal{P}$ in $\mathcal{W}^*$.

Next, we construct a drawing Γ^* of S^* (see Figure 11b).

Lemma 11 *There exists a straight-line plane drawing Γ^* of S^* such that:*

(i) Γ^* *is equivalent to the restriction of* $\mathcal{E}_{C'(f)}$ *to* S^**;*

(ii) δ_f *is represented by* Δ_f*;*

(iii) every simple cycle of S^* *different from* δ_f *is a strictly-convex empty polygon in* Γ^**;*

(iv) all the vertices of S^* *incident to exclusive edges of* $R'(f)$ *and* $B'(f)$ *are on* $\mathcal{P}^*$ *and in the interior of* $\mathcal{W}^*$*; and*

(v) Γ^* *has no intersection with* $\mathcal{H}^*$*.*

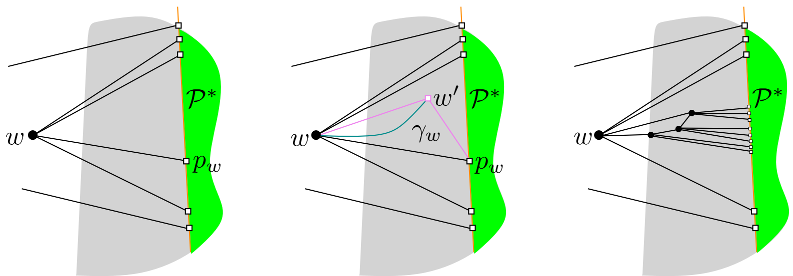
Proof: We construct Γ^* by iteratively drawing 2-connected components of S^*. Every such component is either a simple cycle or an edge, given that S^* is a cactus graph. Recall that, by Property 2, every vertex of S^* incident to an exclusive edge of $R'(f)$ or $B'(f)$ has degree one in S^*.

Initialize Γ^* by drawing straight-line segments from $u_1, \dots, u_k$ to points on $\mathcal{P}^*$. For each $1 \leq i \leq k$, the number of drawn straight-line segments incident to u_i is equal to the number of 2-connected components of S^* containing u_i and different from δ_f; by choosing the endpoints of the segments incident to u_i sufficiently close to p_i on $\mathcal{P}^*$, it can be ensured that all these segments do not cross each other and have empty intersection with $\mathcal{H}^*$. Some drawn segments are *real*, that is, they represent edges of S^*. Some other segments are *dummy*, that is, they represent subgraphs of S^* that still need to be drawn. Straight-line segments appear around each vertex u_i in the order in which the corresponding subgraphs of S^* appear around u_i according to $\mathcal{E}_{C'(f)}$.

Now assume to have a plane straight-line drawing Γ^* of a subgraph D^* of S^* such that the following invariant is satisfied (in addition to the properties in the statement of the lemma).

> Consider the cactus graphs that result from the removal of the edges of D^* from S^*. Each of these graphs that is not a single vertex is represented in Γ^* by a dummy straight-line segment from its only vertex in Γ^* to a point on $\mathcal{P}^*$; further, all these dummy straight-line segments do not cross each other, do not cross any other segment in Γ^*, and have empty intersection with $\mathcal{H}^*$.

Note that the invariant is satisfied by Γ^* after the initialization. Then it suffices to show how the invariant is maintained after drawing in Γ^* a 2-connected component D of S^*, where just one vertex w of D is already in Γ^*, and where a dummy straight-line segment $\overline{wp_w}$ with $p_w \in \mathcal{P}^*$ represents D in Γ^*, as in Figure 12a. Observe that D is a simple cycle, as if it were an edge, it would be represented by a real straight-line segment and not a dummy straight-line



(a) (b) (c)

Figure 12: (a) A dummy straight-line segment $\overline{wp_w}$ representing a component D in Γ^*. (b) Point w' and curve γ_w. (c) Drawing component D in Γ^*.

segment. Consider a point w' arbitrarily close to the midpoint of $\overline{wp_w}$. Draw a strictly-convex curve γ_w inside triangle $\Delta_w = (w, w', p_w)$ connecting w and w', as in Figure 12b. Place the vertices of D on γ_w, in the order they occur along D according to $\mathcal{E}_{C'(f)}$; also, draw the edges of D as straight-line segments, as in Figure 12c. Remove $\overline{wp_w}$ from Γ^*. The polygon representing D is empty, provided that w' is sufficiently close to $\overline{wp_w}$, and strictly-convex, since its vertices lie on a strictly-convex curve. Further, the straight-line segments from the vertices of D to p_w do not cross each other, do not cross any other segment in Γ^*, and have empty intersection with $\mathcal{H}^*$, provided that w' is sufficiently close to $\overline{wp_w}$. Hence, a suitable number of points on $\mathcal{P}^*$ can be chosen, all sufficiently close to p_w so that the straight-line segments between these points and the vertices of D do not cross each other, do not cross any other segment in Γ^*, and have empty intersection with $\mathcal{H}^*$; thus, the invariant is satisfied by the new drawing Γ^*, which concludes the proof. □

The construction of Γ'_f is completed as follows (see Figure 11c). First, we delete u^* and its incident straight-line segments from Γ''_f. Second, we rotate Γ''_f counter-clockwise by an angle of $3\pi/4$. Third, we scale Γ''_f down so that it fits inside a disk D_ε with a suitably small radius $\varepsilon > 0$. Fourth, we place Γ''_f in Γ^* so that D_ε is inside $\mathcal{H}^*$ and is tangent to the half-lines delimiting $\mathcal{W}^*$. Finally, we complete the drawing of the exclusive edges of $R'(f)$ and $B'(f)$ by drawing straight-line segments from their bend-points previously adjacent to u^* to the suitable vertices of S^* on $\mathcal{P}^*$. We have the following.

Lemma 12 *Γ'_f is a* SEFE *of $R'(f)$ and $B'(f)$ with the properties required by Lemma 9, provided that ε is sufficiently small.*

Proof: We first prove that Γ'_f is a SEFE of $R'(f)$ and $B'(f)$. First, vertices and edges of $C'(f)$ have a unique representation in Γ'_f, hence it suffices to prove that the drawings of $R'(f)$ and B'_f in Γ'_f are planar; we will argue about the planarity of the drawing of $R'(f)$, as the one of B'_f can be proved analogously.

By Lemma 10, the drawing of $R''(f)$ in Γ''_f is plane. By Lemma 3, Γ''_f stays plane after all the components different from S^* have been expanded. By Lemma 11, the drawing Γ^* of S^* is plane, as well. Further, Γ''_f and Γ^* do not cross each other, as the former lies in a disk D_ε which is inside $\mathcal{H}^*$, provided that ε is sufficiently small, while the latter does not intersect $\mathcal{H}^*$, by Lemma 11. It remains to argue that the straight-line segments drawn to restore the exclusive edges of $R'(f)$ do not cause crossings.

- First, these segments lie in $\mathcal{H}^*$ if ε is small enough, hence they do not intersect Γ^*.

- Second, they do not intersect red edges in Γ''_f; namely, by Lemma 10 and assuming that the components of $C'(f)$ different from S^* have been expanded in sufficiently small disks, we have that all the red edges in Γ''_f lie to the right of the line ℓ_v with equation $x = 1.5$. After the rotation of Γ''_f by $3\pi/4$ counter-clockwise, Γ''_f is above the rotated line ℓ_v. Thus, it suffices to prove that all the straight-line segments drawn to reconnect the exclusive edges of $R'(f)$ are below or on ℓ_v. Indeed, by Lemmata 10 and 11 each of these segments has one endpoint on ℓ_v and the other endpoint in the interior of $\mathcal{W}^*$; further, ℓ_v is arbitrarily close, depending on the value of ε, to the line delimiting $\mathcal{W}^*$ with slope $5\pi/4$. Hence, each straight-line segment drawn to reconnect an exclusive edge of $R'(f)$ has one end-point on ℓ_v and one end-point below it, provided that ε is sufficiently small.

- Third, the straight-line segments drawn to reconnect the exclusive edges of $R'(f)$ do not cross each other, since the clockwise order in which the edges of $R'(f)$ are incident to u^* (which by Lemma 10 is also the left-to-right order in which the endpoints of the deleted red straight-line segments appear on ℓ_v after the rotation) coincides with the counter-clockwise order in which they are incident to vertices in S^* (which is also the left-to-right order in which these vertices appear along $\mathcal{P}^*$).

The bound on the number of bends in Γ'_f follows from the corresponding bound for Γ''_f in Lemma 10 and from the fact that, when a component of $C'(f)$ is expanded, no new bends are introduced on the exclusive edges. In particular, the exclusive edges incident to one or two vertices in S^* have respectively one or two straight-line segments in Γ'_f they did not have in Γ''_f. However, in turn, they lost one or two straight-line segments in Γ'_f they used to have in Γ''_f, namely those incident to u^*.

The bound on the number of crossings is established as in Theorem 2. Consider any two exclusive edges e'_r of $R'(f)$ and e'_b of $B'(f)$. By Lemma 10, the corresponding edges e''_r in $R''(f)$ and e''_b in $B''(f)$ cross at most sixteen times in Γ''_f, also counting adjacencies. While the expansions might introduce proper crossings between e'_r and e'_b, they only do so in correspondence of an adjacency between e''_r and e''_b; hence e'_r and e'_b cross at most sixteen times in Γ'_f.

Finally, the properties that the edges of $C'(f)$ are straight, that Γ'_f restricted to $R'(f)$, $B'(f)$, and $C'(f)$ is equivalent to $\mathcal{E}_{R'(f)}$, $\mathcal{E}_{B'(f)}$, and $\mathcal{E}_{C'(f)}$, respectively, that δ_f is represented by Δ_f, and that every simple cycle of $C'(f)$ different from δ_f is represented in Γ'_f by a strictly-convex empty polygon have been explicitly ensured while performing the construction. □

Lemma 12 concludes the proof of Lemma 9. Next, plug Γ'_f in Γ'', so that they coincide along Δ_f, obtaining a drawing Γ'''. As in Case 1 and relying on Lemma 9, it is easily shown that Properties BENDS AND CROSSINGS, EMBEDDING, and POLYGONS are satisfied by Γ''', thus completing the discussion of Case 2. We get the following.

Theorem 3 *Let R and B be two planar graphs. If there exists a* SEFE *of R and B, then there also exists a* SEFE *in which every edge is a polygonal curve with at most six bends, every common edge is a straight-line segment, and every two exclusive edges cross at most sixteen times.*

Proof: By Property BENDS AND CROSSINGS, every drawing Γ'' constructed by initializing $R'' = B'' = C'' = \delta^*$ and $\Gamma'' = \Delta^*$, and by then repeatedly applying Case 1 or Case 2 described above is such that every exclusive edge is a polygonal curve with at most four bends, every common edge is a straight-line segment, and every two exclusive edges cross at most sixteen times. Eventually $\Gamma'' = \Gamma'$ is a SEFE of R' and B'. By Lemma 8, the drawing obtained from Γ' by removing vertices and edges not in R and B is a SEFE of R and B satisfying the required properties. □

6 Conclusions

In this paper we proved upper bounds for the number of bends per edge and the number of crossings required to realize a SEFE with polygonal curves as edges. While the bound on the number of bends per edge we presented for tree-tree pairs is tight, there is room for improvement for pairs of planar graphs, as the best known lower bound [9] only states that one bend per edge might be needed. We suspect that our upper bound could be improved by designing an algorithm that constructs simultaneous embeddings of two planar multigraphs with fixed planar embedding with less than four bends per edge. A related interesting problem is to determine how many bends per edge are needed to construct a simultaneous embedding (without fixed edges) of pairs of (simple) planar graphs. The best known upper bound is two [12, 13, 26] and the best known lower bound is one [20]. As a final research direction, we mention the problem of constructing SEFEs of pairs of planar graphs in polynomial area, while matching our bounds for the number of bends and crossings.

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