Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 22, no. 2, pp. 273–294 (2018)
DOI: 10.7155/jgaa.00470

Intersection Graphs of Rays and Grounded Segments

Jean Cardinal[1] *Stefan Felsner*[2] *Tillmann Miltzow*[1,3]
Casey Tompkins[4] *Birgit Vogtenhuber*[5]

[1]Université libre de Bruxelles (ULB)
[2]Institut für Mathematik, Technische Universität Berlin (TU)
[3]Institute for Computer Science and Control,
Hungarian Academy of Sciences (MTA SZTAKI).
[4]Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
[5]Institute of Software Technology, Graz University of Technology

Abstract

We consider several classes of intersection graphs of line segments in the plane and prove new equality and separation results between those classes. In particular, we show that:

- intersection graphs of grounded segments and intersection graphs of downward rays form the same graph class,
- not every intersection graph of rays is an intersection graph of downward rays, and
- not every outer segment graph is an intersection graph of rays.

The first result answers an open problem posed by Cabello and Jejčič. The third result confirms a conjecture by Cabello. We thereby completely elucidate the remaining open questions on the containment relations between these classes of segment graphs. We further characterize the complexity of the recognition problems for the classes of outer segment, grounded segment, and ray intersection graphs. We prove that these recognition problems are complete for the existential theory of the reals. This holds even if a 1-string realization is given as additional input.

Submitted:	Reviewed:	Revised:	Accepted:	Final:
October 2017	February 2018	March 2018	May 2018	June 2018
		Published:		
		August 2018		
	Article type:		Communicated by:	
	Regular paper		C. D. Tóth	

Stefan Felsner is partially supported by DFG grant FE-340/11-1. Tillmann Miltzow is supported by the ERC grant PARAMTIGHT: "Parameterized complexity and the search for tight complexity results", no. 280152.

E-mail addresses: jcardin@ulb.ac.be (Jean Cardinal) felsner@math.tu-berlin.de (Stefan Felsner) t.miltzow@gmail.com (Tillmann Miltzow) ctompkins496@gmail.com (Casey Tompkins) bvogt@ist.tugraz.at (Birgit Vogtenhuber)

1 Introduction

Intersection graphs encode the intersection relation between objects in a collection. More precisely, given a collection $\mathcal{A}$ of sets, the induced intersection graph has the collection $\mathcal{A}$ as the set of vertices, and two vertices $A, B \in \mathcal{A}$ are adjacent whenever $A \cap B \neq \emptyset$. Intersection graphs have drawn considerable attention in the past thirty years, to the point of constituting a whole subfield of graph theory (see, for instance, the book from McKee and McMorris [20]). The roots of this subfield can be traced back to the properties of interval graphs — intersection graphs of intervals on a line — and their role in the discovery of the linear structure of bacterial genes by Benzer in 1959 [1].

We consider *geometric* intersection graphs, that is, intersection graphs of simple geometric objects in the plane, such as curves, disks, or segments. While early investigations of such graphs are a half-century old [28], the modern theory of geometric intersection graphs was established in the 1990s by Kratochvíl [14, 15], and Kratochvíl and Matoušek [16, 17]. They introduced several classes of intersection graphs that are the topic of this paper. Geometric intersection graphs are now ubiquitous in discrete and computational geometry, and deep connections to other fields such as complexity theory [19, 24, 25] and order dimension theory [7, 8, 10] have been established.

We will focus on the following classes of intersection graphs, most of which are subclasses of intersection graphs of line segments in the plane, or *segment (intersection) graphs*. In this paper, all geometric objects we consider lie in the plane.

Grounded Segment Graphs. Given a *grounding line* ℓ in the plane, we call a segment s in the plane a *grounded segment* if one of its endpoints, called the base point, is on ℓ and the interior of s is above ℓ. A graph G is a *grounded segment graph* if it is the intersection graph of a collection of grounded segments (w.r.t. the same grounding line ℓ).

Outer Segment Graphs. Given a *grounding circle* $\mathcal{C}$ in the plane, a segment s in the plane is called an *outer segment* if exactly one of its endpoints, called the base point, is on $\mathcal{C}$ and the interior of s is inside $\mathcal{C}$. A graph G is an *outer segment graph* if it is the intersection graph of a collection of outer segments (w.r.t. the same grounding circle $\mathcal{C}$).

Ray Graphs and Downward Ray Graphs. A graph G is a *ray graph* if it is the intersection graph of rays (halflines) in the plane. A ray r in the plane is called a *downward ray* if its apex is above all other points of r. A graph G is a *downward ray graph* if it is the intersection graph of a collection of downward rays. It is not difficult to see that every ray graph is also an outer segment graph: consider a circle C that contains all intersections in its interior, make C the grounding circle and restrict every ray to the interior of C. Similarly, one can check that every downward ray graph is a grounded segment graph.

String Graphs. *String graphs* are defined as intersection graphs of collections of Jordan arcs in the plane with no three intersecting in the same point. A Jordan arc in the plane is the image of an injective continuous map of a closed interval into the plane. We consider here *1-string graphs*, defined as intersection graphs of strings that pairwise intersect at most once. In particular, we define *outer 1-string graphs* and *grounded 1-string graphs* in the same way as for segments.

In a recent paper, Cabello and Jejčič initiated a comprehensive study aiming at refining our understanding of the containment relations between classes of geometric intersection graphs involving segments, disks, and strings [3]. They introduced and solved many questions about the containment relations between various classes. In particular, they proved proper containment between intersection graphs of segments with k or $k+1$ distinct lengths, intersection graphs of disks with k or $k+1$ distinct radii, and intersection graphs of outer strings and outer segments. In their conclusion [3], they left two natural questions open:

- Is the class of ray graphs a proper subclass of the class of outer segment graphs?
- Is the class of downward ray graphs a proper subclass of the class of grounded segment graphs?

In this contribution, we answer the first question in the positive, thereby proving a conjecture of Cabello. We also give a negative answer to the second question by showing that downward rays and grounded segments yield the same class of intersection graphs. We henceforth completely settle the remaining open questions on the containment relations between these classes of segment graphs. We summarize the complete containment relationship in Theorem 1.

Theorem 1 *The following containment relations of intersection graph classes hold:*

1. *grounded segment graphs = downward ray graphs,*
2. *downward ray graphs* $\subsetneq$ *ray graphs,*
3. *ray graphs* $\subsetneq$ *outer segment graphs,*
4. *outer segment graphs* $\subsetneq$ *outer* 1*-string graphs, and*
5. *outer* 1*-string graphs = grounded* 1*-string graphs.*

Note that Item 4 of Theorem 1 was proved already by Cabello and Jejčič and that Item 5 can be seen as folklore. A schematic description of the established inclusion relations between the graph classes we consider is given in Figure 1.

For the sake of completeness, we reprove all inclusions and equalities of Theorem 1 in Section 2. In Section 5, we show that downward ray graphs are a *proper* subclass of ray graphs and ray graphs are a *proper* subclass of outer segment graphs. See Theorem 3 and Theorem 4.

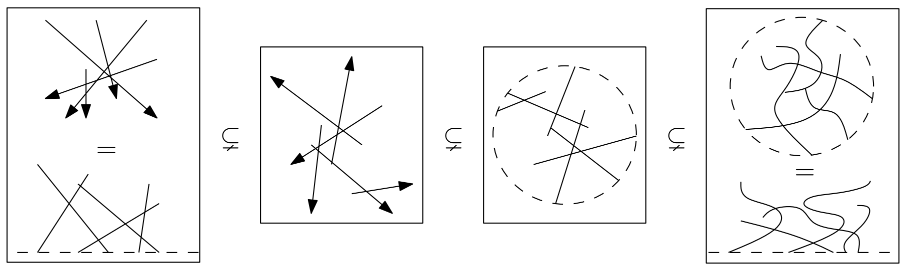

Figure 1: Schematic description of Theorem 1.

Given a representation R of a graph G in one of the ways defined above, we get in a natural way an ordering $\sigma(R)$ of all the vertices. Note that a given graph G might be realized with two different representations $R \neq R'$, which introduce two different orderings $\sigma \neq \sigma'$. Conversely, given a graph G and an ordering π, we can enforce that any representation R of G induces the ordering π.

The Complexity Class $\exists\mathbb{R}$ and the Stretchability Problem The complexity class $\exists\mathbb{R}$ is the collection of decision problems that are polynomial-time equivalent to deciding the truth of sentences in the first-order theory of the reals of the form:

$$\exists x_1 \exists x_2 \dots \exists x_n F(x_1, x_2, \dots, x_n),$$

where F is a quantifier-free formula involving inequalities and equalities of polynomials in the real variables x_i. This complexity class can be understood as a "real" analogue of NP. It can easily be seen to contain NP, and is known to be contained in PSPACE [4].

In recent years, this complexity class revealed itself most useful for characterizing the complexity of realizability problems in computational geometry. A standard example is the *pseudoline stretchability problem.*

Matoušek [18, page 132] defines an *arrangement of pseudolines* as a finite collection of curves in the plane that satisfy the following conditions:

(i) Each curve is x-monotone and unbounded in both directions.

(ii) Every two of the curves intersect in exactly one point, and they cross at the intersection.

In the stretchability problem, one is given the combinatorial structure of an arrangement of pseudolines in the plane as input, and is asked whether the same combinatorial structure can be realized by an arrangement of *straight lines.* If this is the case, then we say that the arrangement is *stretchable.* This structure can for instance be given in the form of a set of n *local sequences*: the left-to-right order of the intersections of each line with the $n-1$ others. Equivalently, the input is the underlying rank-3 oriented matroid. The stretchability problem is known to be $\exists\mathbb{R}$-complete [26]. We refer the reader to the surveys by Schaefer [24], Matoušek [19], and Cardinal [5] for further details.

Computational Complexity Questions Given a graph class $\mathcal{G}$, we define **Recognition**($\mathcal{G}$) as the following decision problem:

Recognition ($\mathcal{G}$)

Input: A graph $G = (V, E)$.
Question: Does G belong to the graph class $\mathcal{G}$?

Potentially the recognition problem could become easier if we have some additional information. In our case it is natural to ask if a given outer 1-string representation of a graph G has an outer segment representation. The same goes for grounded 1-strings and grounded segments. Finally, we will consider outer 1-strings and rays. Formally, we define the decision problem **Stretchability**($\mathcal{G}, \mathcal{F}$) as follows.

Stretchability ($\mathcal{G}, \mathcal{F}$)

Input: A graph $G = (V, E)$ and representation R that shows that G belongs to $\mathcal{F}$.
Question: Does G belong to the graph class $\mathcal{G}$?

Note that we need to assume that $\mathcal{F}$ is a graph class defined by intersections of certain objects.

We also complete the picture by giving computational hardness results on recognition and stretchability questions by proving the following theorem.

Theorem 2 *The following problems are $\exists\mathbb{R}$-complete:*

- ***Recognition***(*grounded segment graphs*) *and*
  Stretchability(*grounded segment graphs, grounded* 1*-string graphs*)*,*
- ***Recognition***(*ray graphs*) *and*
  Stretchability(*ray graphs, outer* 1*-string graphs*)*, and*
- ***Recognition***(*outer segment graphs*) *and*
  Stretchability(*outer segment graphs, outer* 1*-string graphs*)*.*

We want to point out that all statements of Theorem 2 are proven in one simple and unified way. This uses heavily the complete chain of containment of the graph classes and the intrinsic similarity of all considered graph classes. A highlight of Theorem 2 is certainly the $\exists\mathbb{R}$-completeness of the recognition problem for ray intersection graphs. Note that this strengthens the result of Cabello and Jejčič on the separation between outer 1-string and outer segment graphs.

The main idea of the proof of Theorem 2 is a reduction from stretchability. One important tool is the order forcing lemma, which we already mentioned above. In addition to that we need a new ingredient. In this case, we introduce in a simple way so-called, *probes*, which enforce that the order of intersections for certain segments are in an order that is prescribed by the given pseudoline arrangement.

Previous Work and Motivation. The understanding of the inclusion properties and the complexity of the recognition problem for classes of geometric intersection graphs have been the topic of numerous previous works.

Early investigations of string graphs date back to Sinden [28], and Ehrlich, Even, and Tarjan [9]. Kratochvíl [14] initiated a systematic study of string graphs, including the complexity-theoretic aspects [15]. It is only relatively recently, however, that the recognition problem for string graphs has been identified as NP-complete [25]. NP membership is far from obvious, given that there exist string graphs requiring exponential-size representations [16].

Intersection graphs of line segments were extensively studied by Kratochvíl and Matoušek [17]. In particular, they proved that the recognition of such graphs was complete for the existential theory of the reals. A key construction used in their proof is the *Order-forcing Lemma*, which permits the embedding of pseudoline arrangements as segment representations of graphs. Some of our constructions can be seen as extensions of the Order-forcing Lemma to grounded and outer segment representations.

Outer segment graphs form a natural subclass of outer string graphs as defined by Kratochvíl [14]. They also naturally generalize the class of *circle graphs*, which are intersection graphs of chords of a circle [22].

A recent milestone in the field of segment intersection graphs is the proof of Scheinerman's conjecture by Chalopin and Gonçalves [6], stating that planar graphs form a subclass of segment graphs. It is also known that outerplanar graphs form a proper subclass of circle graphs [30], hence of outer segment graphs. Cabello and Jejčič [3] proved that a graph is outerplanar if and only if its 1-subdivision is an outer segment graph.

Intersection graphs of rays in two directions have been studied by Soto and Telha [29]. They show connections with the jump number of some posets and hitting sets of rectangles. The class has been further studied by Shrestha et al. [27], and Mustaţă et al. [21]. The results include polynomial-time recognition and isomorphism algorithms. This is in contrast with our hardness result for arbitrary ray graphs.

Properties of the chromatic number of geometric intersection graphs have been studied as well. For instance, Rok and Walczak proved that outer string graphs are χ-bounded [23], and Kostochka and Nešetřil [12, 13] studied the chromatic number of ray graphs in terms of the girth and the clique number.

The complexity of the maximum clique and independent set problems on classes of segment intersection graphs is also a central topic of study. It has been shown recently, for instance, that the maximum clique problem is NP-hard on ray graphs [2], and that the maximum independent set problem is polynomial-time tractable on outer segment graphs [11].

Organization of the Paper. In the next section, we give some basic definitions and observations. We also provide a short proof of the equality between the classes of downward ray and of grounded segment graphs.

In Section 3, we introduce the *Cycle Lemma*, a construction that will allow us to control the order of the slopes of the rays in a representation of a ray graph, and the order in which the segments are attached to the grounding line or circle in representations of grounded segment and outer segment graphs.

In Section 4, we show how to use the Cycle Lemma to encode the pseudoline stretchability problem in the recognition problem for outer segment, grounded segment, and ray graphs. We thereby prove that those problems are complete for the existential theory of the reals.

Finally, in Section 5, we establish two new separation results. First, we prove that ray graphs form a proper subclass of outer segment graphs, proving Cabello's conjecture. Then we prove that downward ray graphs form a proper subclass of ray graphs.

2 Preliminaries

We first give a short proof of the equality between the classes of downward ray and grounded segment graphs, thereby answering Cabello and Jejčič's second question. The proof is illustrated in Figure 2.

Lemma 1 (Downward Ray Graphs = Grounded Segment Graphs) *A graph G can be represented as a grounded segment graph if and only if it can be represented by downward rays.*

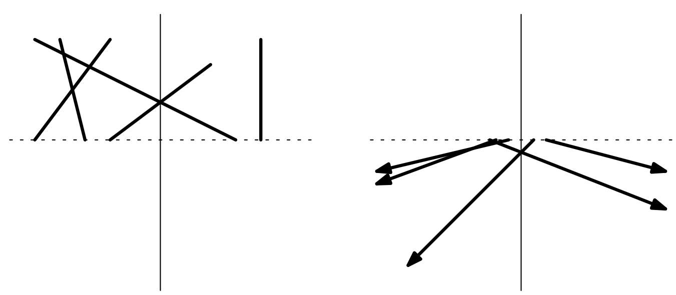

Figure 2: Grounded segments and downward rays.

Proof: Consider a coordinate system where the grounding line is the x-axis, and take the projective transformation defined in homogeneous coordinates by

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} x \\ -1 \\ y \end{pmatrix}.$$

This projective transformation is a bijective mapping from the projective plane to itself, which maps grounded segments to downward rays. In the plane, it can be seen as mapping the points (x, y) with $y > 0$ to $(x/y, -1/y)$. Projective transformations preserve the incidence structure and straightness. Thus the equivalence of the graph classes follows. □

Lemma 2 (Ray Graph Characterization) *A graph G can be represented as an outer segment graph with all intersections of the supporting lines inside the grounding circle $\mathcal{C}$ if and only if it can be represented by rays.*

Proof: See Figure 3 for an illustration of the following.

(⇐) Let R be a representation of G by rays, and let $\mathcal{L}$ be the set of the lines extending all involved rays. Then there exists a circle $\mathcal{C}$ that contains all the intersections of $\mathcal{L}$ and at least some part of each ray. We define a representation R' of G as outer segment representation by restricting each ray to the inside of $\mathcal{C}$. It is easy to see that this indeed is a representation of G with the desired property.

(⇒) Let R be a representation of G by outer segments with all intersections of the supporting lines inside the grounding circle $\mathcal{C}$. We define a set of rays by just extending each segment at its base point on the grounding circle $\mathcal{C}$ to a ray. If two segments intersected before, then the corresponding rays will intersect as well trivially. Moreover, by the assumption that all the line extensions intersect inside $\mathcal{C}$, it follows that the rays will not intersect outside $\mathcal{C}$s, and hence the cooresponding ray graph is a representation of G. □

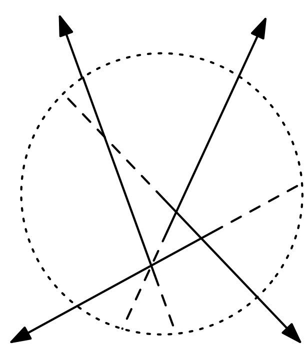

Figure 3: Rays and outer segments.

Note that it is tempting to try to find a projective transformation that maps the unit circle S^1 to infinity in a way that outer segments become rays. As we will show later, outer segments and rays represent different graph classes. Thus such a mapping is impossible. With the help of Möbius transformations it is possible to find a mapping that maps the unit circle S^1 to infinity. However, outer segments then become connected parts of hyperbolas instead of straight-line rays.

For the following lemma, recall that we define *grounded* 1-*string graphs* and *outer* 1-*string graphs* in an analogous way to the corresponding segment graphs by replacing segments by 1-strings.

Lemma 3 (Grounded 1-String Graphs = Outer 1-String Graphs) *A graph G can be represented as a grounded* 1*-string graph if and only if it can be represented as an outer* 1*-string graph.*

Proof: See Figure 4 for an illustration of this proof.

($\Rightarrow$) Let R be a representation of G by grounded 1-strings with grounding line ℓ. Take a large circle $\mathcal{C}$ that completely contains R and extend the 1-strings perpendicularly from the grounding point on ℓ to the opposite side of ℓ until they meet the circle $\mathcal{C}$. This procedure yields an outer 1-string representation with grounding circle $\mathcal{C}$ and the same incidences as R, hence an outer 1-string representation of G.

($\Leftarrow$) Let R be a representation of G by outer 1-strings grounded on a circle $\mathcal{C}$. Let ℓ be a horizontal line below $\mathcal{C}$. Extend any 1-string whose grounding point is on the bottom half of $\mathcal{C}$ with a vertical line segment to ℓ. Extend any 1-string whose grounding point is on the top half of $\mathcal{C}$ with a horizontal segment followed by a vertical segment from $\mathcal{C}$ to the line ℓ. This procedure clearly does not alter any incidences. Thus it provides a grounded segment representation of G. $\square$

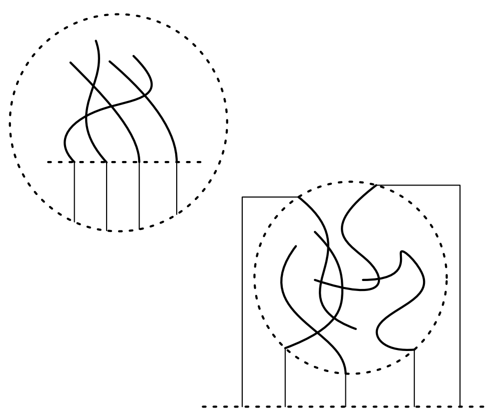

Figure 4: Outer 1-strings and grounded 1-strings.

Ordered Representations. Given a graph G and a permutation π of the vertices, we say that a grounded (segment or string) representation of G is π-*ordered* if the base points of the cooresponding segments or strings are in the order of π on the grounding line, up to inversion and cyclic shifts. In the same fashion, we define π*-ordered* for outer (segment or string) representations and (downward) ray representations, where rays are ordered by their angles with the horizontal axis.

3 Cycle Lemma

For some of our constructions, we would like to force that the segments or strings representing the vertices of a graph appear in a specified order on the grounding line or circle. More exactly, we would like to force the representation to be π-ordered for some given permutation π. To this end, we first study some properties of the representation of cycles, which in turn will help us to enforce this order.

Given a graph $G = (V, E)$ on n vertices $V = \{v_1, \ldots, v_n\}$ and a permutation π of the vertices of G, we define the *order forcing graph* G^π as follows. The vertices $V(G^\pi)$ are defined by $V \cup \{1, \ldots, 2n^2\}$ and the edges $E(G^\pi)$ are defined by $E \cup \{\, (2in, v_{\pi(i)}) \mid i = 1, \ldots, n \,\} \cup \{\, (i, i+1) \mid i = 1, \ldots, 2n^2 \,\}$ (here, for convenience, we consider addition modulo $2n^2$ so that $2n^2 + 1 = 1$). The definition is illustrated on Figure 5.

For the sake of simplicity, we think of π as being the identity and the vertices as being indexed in the correct way. The vertices of G are called *relevant*, and the additional vertices of G^π are called *cycle vertices*. Note that on the cycle, the distance between any two cycle vertices u, v that are adjacent to different relevant vertices is at least $2n$.

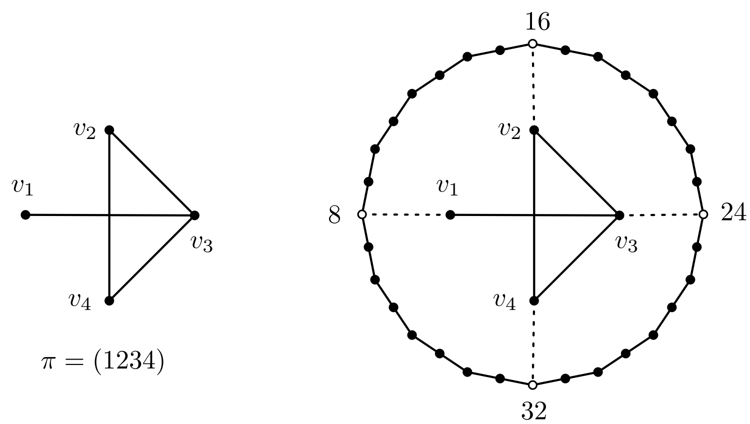

Figure 5: Illustration of the definition of order forcing graphs.

Lemma 4 (Cycle Lemma) *Let G be a graph and π be a permutation of the vertices of G. Then there exists a π-ordered representation of G if and only if there exists a representation of G^π. This is true for the following graph classes: grounded segment graphs, ray graphs, outer segment graphs, and outer 1-string graphs.*

Note that for the case that $|V(G)| \leq 3$ this statement is trivial, as it can be easily checked that in these finitely many cases both graphs can always be realized. Thus from now on, we assume that $|V(G)| \geq 4$.

Before proving Lemma 4 we first study the representations of cycles. Let $C = 1, 2, 3, \ldots, n$ be a cycle of length n and R be a 1-string representation of C. Then each string i has exactly two crossings, namely, one with string $(i-1)$ and one with string $(i+1)$. The part of i between the two intersections is called *central part* of i and denoted by z_i. The intersection points are denoted by $p_{i,i-1}$ and $p_{i,i+1}$. We use this notation throughout this section.

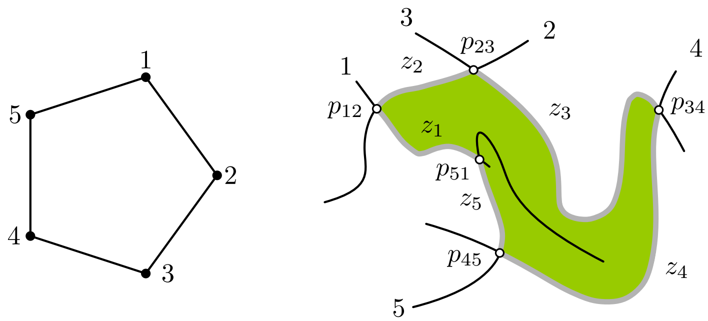

Figure 6: A 1-string representation of a 5-cycle.

Lemma 5 *Let C be a cycle and R be a 1-string representation of C. The union of all central parts of all the 1-strings of R forms a Jordan curve, which we denote by $J(C)$. This also holds in case that C is an induced subgraph of some other graph G.*

Proof: Using the above notation, the curve can be explicitly given as:

$$J(C) := p_{12}, z_2, p_{23}, z_3, \ldots, z_n, p_{n1}, z_1.$$

Obviously, $J(C)$ is a continuous curve. Moreover, as C is a cycle or an induced cycle of some other graph G, there are no further crossings between the strings of R and hence $J(C)$ is indeed a Jordan curve. □

Lemma 6 *Let C be an induced cycle of the graph G and R be an outer 1-string representation of G. Further, let $a, b \notin V(C)$ be two adjacent vertices, which are respectively adjacent to u_a and u_b in $V(C)$ with $dist(u_a, u_b) \geq 4$ on the cycle, and to no other vertex of C. Then a must intersect the central part of u_a, b must intersect the central part of u_b, and a and b must intersect in the interior of $J(C)$.*

Proof: Let $i \in V(C)$ be an outer 1-string. We denote by start(i) the portion of i between its base point and the first point on the central part, which we denote by p_i. Given three distinct points p, q, r on $J(C)$, we denote by path(p, q, r) the portion of $J(C)$ bounded by p and q and containing r. Similarly, let p, q, r be three distinct points on the grounding circle. Then there exists a unique portion

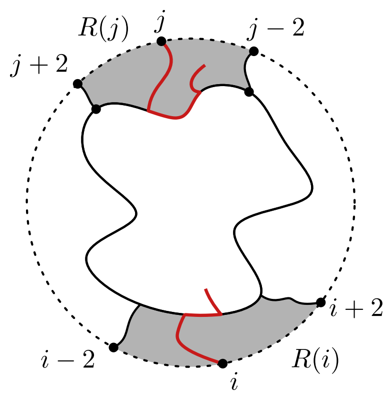

Figure 7: Illustration of Lemma 6.

circle(p, q, r) of the grounding circle bounded by p and q and containing r. For each $i \in V(C)$, we consider the region $R(i)$ bounded by the following four curves:

$$\text{start}(i-2)\ ,\ \text{path}(p_{i-2}, p_{i+2}, p_i)\ ,\ \text{start}(i+2)\ ,\ \text{circle}(i-2, i+2, i).$$

We summarize a few useful facts on these regions.

1. String i is contained in the union of the region $R(i)$ and the interior of $J(C)$.
2. If $\text{dist}(i, j) \geq 4$, then $R(i)$ and $R(j)$ are interior disjoint and $i \cap R_j = \emptyset$.
3. If $v \notin V(C)$ is adjacent to $i \in V(C)$ but not adjacent to any other $j \in V(C)$, then the base point of v must be inside $R(i)$.

The first statement follows from the fact that i is disjoint from $i-2$ and $i+2$, as C is an induced cycle. The second statement can be derived from the definition of $R(i)$ as follows. The two regions $R(i)$ and $R(j)$ define some closed Jordan curve J. As C is an induced cycle, the boundaries of those regions cannot cross because this would produce a forbidden adjacency in C. The last statement follows from the fact that there is no way to reach the central part or $R(i)$ in the case that v does not start in $R(i)$. Suppose for the purpose of contradiction that v does not start in $R(i)$. Then it must start in some other $R(j)$ with $\text{dist}(i, j) \geq 4$, as $R(0), R(4), R(8), \ldots$ cover the outer circle. Furthermore, v is not adjacent to any vertex defining the boundary of $R(j)$, by assumption. Thus v is completely contained in $R(j)$. By Fact 1, string i is completely outside of $R(j)$. Thus v and i do not intersect — contradiction.

We can now complete the proof. The outer 1-string a must have its base point in $R(u_a)$, and b must have its base point in $R(u_b)$, by Fact 3. These two regions have disjoint interiors by Fact 2 and do not have a common boundary formed by any part of u_a or u_b. Hence, as a and b are not allowed to cross any other string of C, they can only intersect in the interior of $J(C)$. □

See Figure 7 for an illustration. We are now ready to prove our main lemma.

Proof of Lemma 4: (⇒) Let R be an ordered representation of G with respect to π. We have to construct a representation R^π of the graph G^π.

We start with the case of outer 1-string graphs. We may assume w.l.o.g. that every string in R intersects its grounding circle exactly in ints base point. Then for any pair u and v of segments that are consecutive w.r.t. π, there is a connected region E incident to the base points of both of them and to the part of the grounding circle between them. For every string s of R, we add a tiny outer segment t_s close to the base point of s that properly crosses s and does not cross any other string of R. It is straightforward to see that for every pair u, v of strings of R that are consecutive w.r.t. π, the segments t_u and t_v can be connected via a path of arbitrarily many strings inside E that does not interfer with the remaining string representation; see the top part of Figure 8 for an illustration.

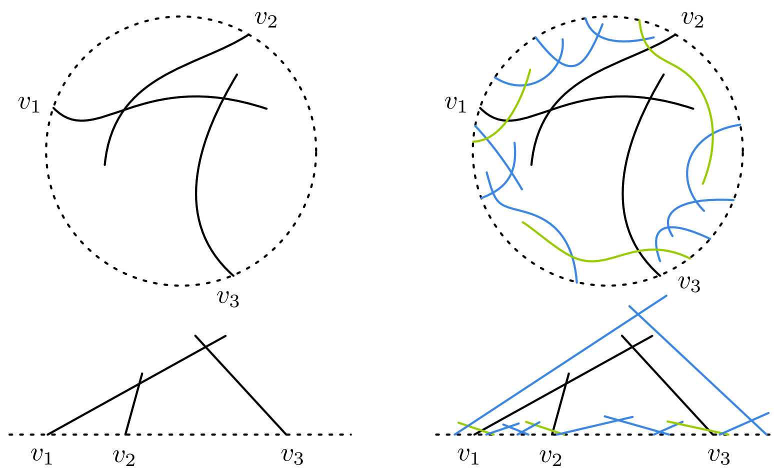

Figure 8: Representations of order forcing graphs for ordered representations of outer string graphs and grounded segment graphs.

For the case of grounded segment graphs, the line of argument is analogous to the one for outer 1-string graphs, with the exception that the paths now consist of grounded segments along the grounding line ℓ. Further, the path between the small segment for the first and the small segment for the last vertex in the order π does not follow the grounding line ℓ but instead surrounds the original grounded segment representation. Obviously, this is feasible with a path having at least 3 segments; see the bottom part of Figure 8 for an example.

Next, we proceed with the case of outer segment graphs, which is more involved. Let R be an ordered outer segment representation of G with respect to π with grounding circle $\mathcal{C}$. Modify R such that each segment stops at its last intersection point. Further, any segment that does not intersect other segments is redrawin such that it hits some other segment. Let $u, v \in V(G)$ be two segments that are successive in the order of π, that is, the base points u and v are consecutive on the grounding circle $\mathcal{C}$. Further let d be the center of $\mathcal{C}$. We denote by E the region inside $\mathcal{C}$, incident to u and v, and outside the convex hull of all the truncated segments plus d; see the illustration on the left side of of Figure 9. In order to construct R^π, we will add the segments representing the cycle vertices to R. As a first step, we add for each segment s of R a tiny segment t_s that crosses s very close to the base point of s, does not cross any other segments, and has both endpoints on the grounding circle $\mathcal{C}$. To close the cycle, we add, for every pair $u, v \in V(G)$ of two segments that are successive in the order of π, a path inside E connecting t_u and t_v. To this end, note that the boundary of E has at most $n/2+1$ reflex points $r_1, \ldots, r_k$, where each reflex point comes from a common endpoint of two truncated segments or the center d of $\mathcal{C}$. We extend k rays from the center d of $\mathcal{C}$ through $r_1, \ldots, r_k$. This divides E in at most $k+1$ convex regions. Note that $k+1 \le n/2+2 \le 2n-1$, for $n \ge 4$. Recall that on the added cycle, the distance between any two cycle vertices a, b which are adjacent to different relevant vertices is at least $2n$. It is easy to see that we can place one grounded segment into each region such that they form the desired path from t_u to t_v without intersecting any other segment from R. In order to obtain enough segments on the path, it might be necessary to use several segments inside one region.

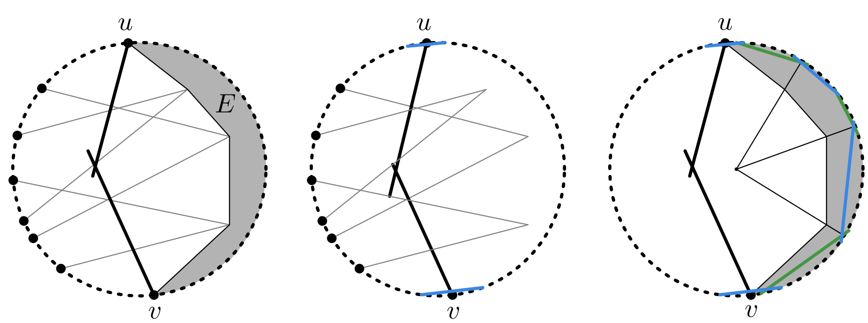

Figure 9: Representation of G^π as outer segment graph.

Now we show the statement for ray graphs, see Figure 10. We start with a representation R of our ordered ray graph of G with respect to π. Let D be a sufficiently large disk that contains all ray starting points as well as all intersections among all the rays and let ∂D be the boundary of D. For each ray r we define ℓ_r to be the line orthogonal to r through the unique point $\partial D \cap r$. Note that ℓ_r is usually not tangent to ∂D. However, when the radius of D goes to infinity and the center of D remains unchanged, then the angle between ℓ_r and ∂D converges to $\pi/2$. Hence, by choosing D large enough, we obtain that the collection of all lines ℓ_r defined in this way determine a *convex* polygonal region P in which each ℓ_r is the supporting line of an edge of P. It can happen that P is unbounded, if all rays are downward for instance. The *convex* polygon Q is defined by adding n sufficiently small edges in the vicinity of each vertex of P.

(Recall that the distance between two consecutive vertices on the cycle is $2n$.) In case P is unbounded, we define Q in a way that it is bounded, by adding an appropriate edge. There are clearly many ways to construct Q. Anyway is fine as long as Q has $n + n^2$ vertices and every $2n$-th edge is intersected by a ray. Now denote with $v_1, \ldots, v_k$ the $k = n + n^2$ orthogonal vectors of the edges $e_1, \ldots, e_k$ of Q in clockwise order. We place at each edge e_i two rays q_i and r_i with slopes v_{i+1} and v_i so that q_i and r_i intersect (their apices being close to the endpoints of the edges and not on any ray of R). This step is illustrated with a regular k-gon Q at the right of Figure 10. It is easy to see that the intersection graph of these rays is a cycle, after a small perturbation. Further each ray of R intersects exactly one of the new rays. The representation R^π of G^π is the union of the rays of R and the newly defined rays.

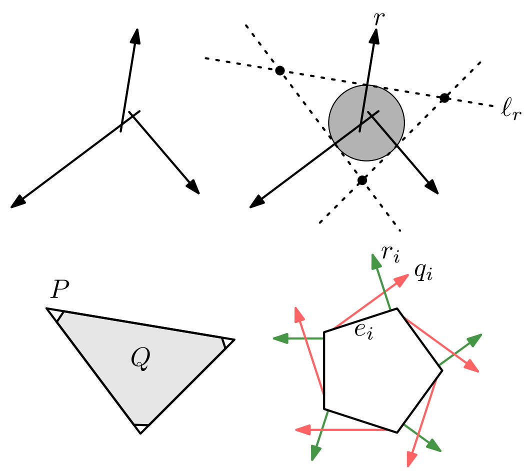

Figure 10: Illustration of the proof of Lemma 4 for rays.

($\Leftarrow$) Recall that we have to show the following. If G^π has a representation, then G also has a π-ordered representation. We show this by considering a representation R^π of G^π. It is clear that R^π restricted to the relevant vertices gives a representation of G. We will show that the vertices $V(G)$ are π-ordered.

It is sufficient to consider outer 1-string graphs. This is easy to see for outer segment graphs as every outer segment representation is *also* an outer 1-string representation. For rays, downward rays and grounded 1-strings, we know how to transform any representation into an outer 1-string representation without altering the order, see Section 2.

By Lemma 6, each relevant outer 1-string adjacent to the circle vertex i is fully contained in the region $R(i)$, as described in the proof of Lemma 6. As all regions $R(2n), R(4n), R(6n), \ldots$ are pairwise disjoint and arranged in this order on the grounding circle, this order is also enforced on the 1-strings of $V(G)$. $\square$

4 Stretchability

The main purpose of this section is to show that the recognition of the graph classes defined above is $\exists\mathbb{R}$-complete. For this we will use Lemma 4 extensively. It is likely that our techniques can be applied to other graph classes as well.

For convenience, we state again the theorem that we prove in this section.

Theorem 2 *The following problems are $\exists\mathbb{R}$-complete:*

- ***Recognition***(*grounded segment graphs*) *and*
  Stretchability(*grounded segment graphs, grounded* 1*-string graphs*)*,*
- ***Recognition***(*ray graphs*) *and*
  Stretchability(*ray graphs, outer* 1*-string graphs*)*, and*
- ***Recognition***(*outer segment graphs*) *and*
  Stretchability(*outer segment graphs, outer* 1*-string graphs*)*.*

Proof: We first show $\exists\mathbb{R}$-membership. Note that each of the straight-line objects we consider can be represented with at most four variables: for segments, we use two variables for each endpoint, and for rays, we use two variables for the apex and two variables for the direction. The condition that two objects intersect can be formulated with constant-degree polynomials in those variables. Hence, each of the problems can be formulated as a sentence in the first-order theory of the reals of the desired form. Note that the given representation of the stretchability instance is correct by assumption and hence need not be verified.

Let us now turn our attention to the $\exists\mathbb{R}$-hardness. It is sufficient to show hardness for the stretchability problems, as the problems can only become easier with additional information. We will reduce from stretchability of pseudoline arrangements. Given a pseudoline arrangement $\mathcal{L}$, we will construct a graph $G_{\mathcal{L}}$ and a permutation π such that the following statements hold:

1. If $\mathcal{L}$ is stretchable, then $G_{\mathcal{L}}$ has a π-ordered representation with *grounded segments.*
2. If $\mathcal{L}$ is not stretchable, then there does not exist a π-ordered representation of $G_{\mathcal{L}}$ as an *outer segment graph.*

By Lemma 4 $G_{\mathcal{L}}$ has a π-ordered representation if and only if $G_{\mathcal{L}}^{\pi}$ has a representation. Recall that we know the following relations for the considered graph classes.

grounded segment graphs $\subseteq$ ray graphs $\subseteq$ outer segment graphs.

Thus, Statement 1 implies that if $\mathcal{L}$ is stretchable then $G_{\mathcal{L}}$ has a π-ordered representation with rays or outer segments. Furthermore, Statement 2 implies that if $\mathcal{L}$ is not stretchable then $G_{\mathcal{L}}$ has neither a π-ordered representation with rays nor with grounded segments.

We start with the construction of $G_{\mathcal{L}}$ and π. Let $\mathcal{L}$ be an arrangement of n pseudolines. Recall that we can represent $\mathcal{L}$ by x-monotone curves. Let ℓ_1 and ℓ_2 be two vertical lines such that all the intersections of $\mathcal{L}$ lie between ℓ_1 and ℓ_2. We cut away the part outside the strip bounded by ℓ_1 and ℓ_2. This gives us a π-ordered grounded 1-string representation $R_{\mathcal{L}}$ with respect to the grounding line ℓ_1.

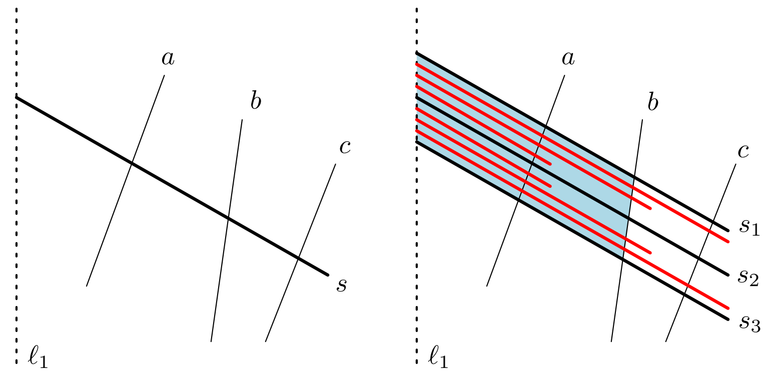

Figure 11: Illustration of Theorem 2: Construction of $G_{\mathcal{L}}$ and its grounded 1-string representation R_G.

Now we replace each string s representing a pseudoline in $\mathcal{L}$ by the following construction (extending π accordingly): We split s into three similar copies s_1, s_2, s_3, shifted vertically by an offset that is chosen sufficiently small so that the three copies intersect the other pseudolines (and their shifted copies) in the same order. For each successive intersection point of s with a pseudoline s' in $\mathcal{L}$, we add a pair of strings grounded on either side of the base point of s_2 and between the base points of s_1 and s_3, intersecting none of s_1, s_2 and s_3. The two strings intersect all the pseudolines of $\mathcal{L}$ that s intersects, up to and including s', in the same order as s does. All the strings for s are pairwise nonintersecting; see Figure 11. We refer to these pairs of strings as *probes*. The probes are meant to enforce the order of the intersections in all π-ordered representations.

We now prove Statement 1. We suppose there is a straight line representation of $\mathcal{L}$, which we denote by $\mathcal{K}$. Again let ℓ_1 and ℓ_2 be two vertical lines such that all intersections of $\mathcal{K}$ are contained in the vertical strip between them. This gives us a collection of grounded segments $R_{\mathcal{K}}$. One can check that the above construction involving probes can be implemented using straight line segments, just as illustrated in Figure 11. Thus, $R_{\mathcal{K}}$ is a π-ordered grounded segment representation of $G_{\mathcal{L}}$, as claimed.

Next, we turn our attention to Statement 2 and suppose that $\mathcal{L}$ is not stretchable. Let us further suppose, for the purpose of contradiction, that we have a π-ordered outer segment representation R'_G of $G_{\mathcal{L}}$. We show that keeping only the middle copy s_2 of each segment s representing a pseudoline of $\mathcal{L}$ in our construction, we obtain a realization of $\mathcal{L}$ with straight lines. For this, we need to prove that the construction of the probes indeed forces the order of the intersections. We consider each such segment s_2 in the grounded 1-string representation R_G of $G_\mathcal{L}$ and orient it from its base point to its other endpoint. Now suppose that in R'_G, there exist strings a and b such that the order of intersections of s_2 with a and b with respect to this orientation does not agree with that of R_G and hence that of the pseudoline arrangement. Assume that in R'_G, s_2 crosses the lines b before a in the left-to-right order, other than in R_G and other than shown in Figure 11. In R'_G, consider the region bounded by the arc of the grounding circle between the base points of s_1 and s_3, and segments from s_1, b, and s_3. Due to the π-orderedness of R'_G, this region is convex and split into two convex sub-regions by s_2. The pair of probes corresponding to the intersection of s_2 and a is completely contained in this region, with one probe in each sub-region. As the line a must intersect those probes, a must enter both sub-regions, thereby intersecting s_2 on the left of b with respect to the chosen orientation, a contradiction. Therefore, the order of the intersections is preserved, and the collection of segments s_2 is a straight line realization of $\mathcal{L}$, a contradiction to the assumption that $\mathcal{L}$ is not stretchable. □

5 Rays and Segments

Theorem 3 (Downward Ray Graphs $\subsetneq$ Ray Graphs) *There are graphs that admit a representation as ray graphs but not as downward ray graphs.*

Proof: We consider the graph G and the permutation π as displayed in Figure 12 (left). We show that G does not have a π-ordered representation as a

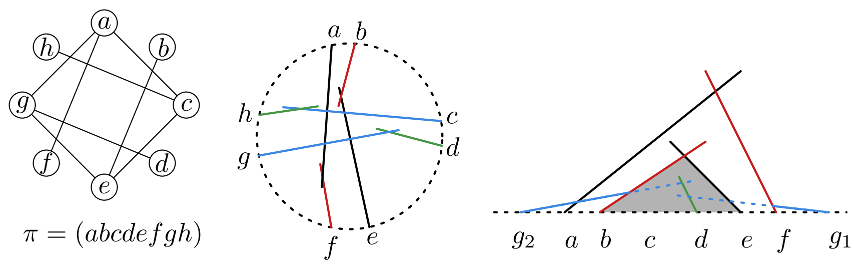

Figure 12: Illustration of Theorem 3: A graph G together with a permutation π of the vertices (left); A π-ordered outer segment representation of G (middle); The segment g cannot enter the gray triangle without intersecting b or f (right).

grounded segment graph, hence G^π has a representation as a ray graph, but not as a grounded segment graph or a downward ray graph; see Lemma 4 and Lemma 1.

Assume for the sake of contradiction that G has a π-ordered representation R^π as a grounded segment graph. As G is rotation symmetric, we can assume without loss of generality that in R^π the base points of the segments a, b, c, d, e, and f are sorted from left to right in this order along ℓ; cf. Figure 12 (right). Consider first the segments a, f, b and e in R^π. As a and f

intersect, they form a triangle Δ_{af} together with ℓ. An according statement holds for b and e with triangle Δ_{be}. Moreover, as none of a and f intersects b or e, and as the base points of b and e lie between the base points of a and f, the triangle Δ_{be}, as well as the whole segments b and e lie completely inside Δ_{af}. Now consider the segment d, which has its base point between b and e. As d does not intersect any of b and e, d lies completely inside Δ_{be}. Finally, consider the segment g which has its base point either to the left of a or to the right of f. The two possibilities are indicated with g_1 and g_2 in Figure 12 (right). On the one hand, g must intersect d and hence enter the triangle Δ_{be}. On the other hand, g is not allowed to intersect any of b and f, a contradiction. □

Theorem 4 (Ray Graphs $\subsetneq$ Outer Segment Graphs) *There are graphs that admit a representation as outer segment graphs but not as ray graphs.*

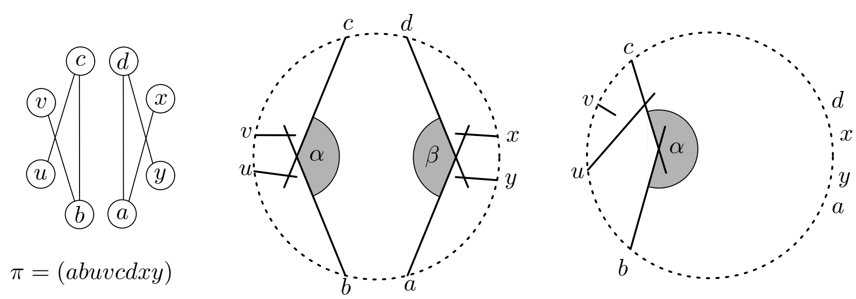

Figure 13: Illustration of Theorem 4. On the left is a graph G together with a permutation π of the vertices displayed. In the middle is a π-ordered outer segment representation of G. The right drawing illustrates that the angles α and β must each be at most 180°.

Proof: Consider the graph G and a permutation π as displayed on the left of Figure 13. We show that G has a π-ordered representation as an outer segment graph, but not as a ray graph. This implies that G^π has a representation as an outer segment graph, but not as a ray graph, see Lemma 4.

Consider any π-ordered outer segment representation of G where grounding points of the segments appear in clockwise order along the grounding circle, for example the one depicted in the middle of Figure 13. Consider the intersection point p_{bc} of the segments b and c and the base points b' and c' of b and c, respectively, and denote as α the angle between $p_{bc}c'$ and $p_{bc}b'$ in clockwise order. Likewise, for the intersection point p_{ad} of the segments a and d and the base points a' and d' of a and d, respectivily, let β be the angle between $p_{ad}a'$ and $p_{ad}d'$ in clockwise order.

We show that both α and β are smaller than 180° in any π-ordered outer segment representation of G. As the two cases are symmetric we show it only for α. Assume $\alpha \geq 180°$ as on the right of Figure 13. Note that if $\alpha \geq 180°$, then the region bounded in clockwise order by $c'p_{bc}$, $p_{bc}b'$, and the part of the grounding circle between c' and b' is convex and the remaining parts of b and c lie outside this region. Hence, u must intersect $c'p_{bc}$ and v must intersect $p_{bc}b'$. However, by the π-orderedness of the representation, if u intersects $c'p_{bc}$ (as in Figure 13) then it blocks v from intersecting b, as v must not intersect u. Likewise, v intersecting b would block u from intersecting c. This shows $\alpha, \beta < 180°$.

As both angles are smaller than 180°, we conclude that either the extensions of a and b or the extensions of c and d must meet outside of the grounding circle (as none of the extensions can intersect any of the segments $b'c'$ and $a'd'$ in the interior of the grounding circle). Recall that we considered any π-ordered outer segment representation of G. By Lemma 2 it holds for every ray graph that there exists at least one representation of G with outer segments such that *all* extensions meet *within* the grounding circle. (The lemma also holds for ordered representations.) Thus there cannot be a π-ordered ray representation of G. $\square$

Acknowledgments.

This work was initiated during the Order & Geometry Workshop organized by Piotr Micek and the second author at the Gułtowy Palace near Poznań, Poland, on September 14-17, 2016. We thank the organizers and attendees, who contributed to an excellent work atmosphere. Some of the problems tackled in this paper were brought to our attention during the workshop by Michał Lasoń. The first author also thanks Sergio Cabello for insightful discussions on these topics. We also want to thank anonymous reviewers for their careful reading and detailed comments. This helped us to improve the write up.

References

[1] S. Benzer. On the topology of the genetic fine structure. *Proceedings of the National Academy of Sciences*, 45(11):1607–1620, 1959. doi:10.1073/pnas.45.11.1607.

[2] S. Cabello, J. Cardinal, and S. Langerman. The clique problem in ray intersection graphs. *Discrete & Computational Geometry*, 50(3):771–783, 2013. doi:10.1007/s00454-013-9538-5.

[3] S. Cabello and M. Jejčič. Refining the hierarchies of classes of geometric intersection graphs. *Electron. J. Combin.*, 24(1):P1.33, 2017.

[4] J. F. Canny. Some algebraic and geometric computations in PSPACE. In *Proc. STOC*, pages 460–467. ACM, 1988. doi:10.1145/62212.62257.

[5] J. Cardinal. Computational geometry column 62. *ACM SIGACT News*, 46(4):69–78, 2015.

[6] J. Chalopin and D. Gonçalves. Every planar graph is the intersection graph of segments in the plane: extended abstract. In *Proc. STOC*, pages 631–638. ACM, 2009. doi:10.1145/1536414.1536500.

[7] S. Chaplick, S. Felsner, U. Hoffmann, and V. Wiechert. Grid intersection graphs and order dimension. *Order*, pages 1–29, 2018. doi:10.1007/s11083-017-9437-0.

[8] S. Chaplick, P. Hell, Y. Otachi, T. Saitoh, and R. Uehara. Intersection dimension of bipartite graphs. In *Proc. TAMC*, volume 8402 of *LNCS*, pages 323–340. Springer, 2014. doi:10.1007/978-3-319-06089-7_23.

[9] G. Ehrlich, S. Even, and R. E. Tarjan. Intersection graphs of curves in the plane. *J. Comb. Theory, Ser. B*, 21(1):8–20, 1976. doi:10.1016/0095-8956(76)90022-8.

[10] S. Felsner. The order dimension of planar maps revisited. *SIAM J. Discrete Math.*, 28(3):1093–1101, 2014. doi:10.1137/130945284.

[11] J. M. Keil, J. S. B. Mitchell, D. Pradhan, and M. Vatshelle. An algorithm for the maximum weight independent set problem on outerstring graphs. *Comput. Geom.*, 60:19–25, 2017. doi:10.1016/j.comgeo.2016.05.001.

[12] A. V. Kostochka and J. Nešetřil. Coloring relatives of intervals on the plane, I: chromatic number versus girth. *Eur. J. Comb.*, 19(1):103–110, 1998. doi:10.1006/eujc.1997.0151.

[13] A. V. Kostochka and J. Nešetřil. Colouring relatives of intervals on the plane, II: intervals and rays in two directions. *Eur. J. Comb.*, 23(1):37–41, 2002. doi:10.1006/eujc.2000.0433.

[14] J. Kratochvíl. String graphs. I. The number of critical nonstring graphs is infinite. *J. Comb. Theory, Ser. B*, 52(1):53–66, 1991. doi:10.1016/0095-8956(91)90090-7.

[15] J. Kratochvíl. String graphs. II. Recognizing string graphs is NP-hard. *J. Comb. Theory, Ser. B*, 52(1):67–78, 1991. doi:10.1016/0095-8956(91)90091-W.

[16] J. Kratochvíl and J. Matoušek. String graphs requiring exponential representations. *J. Comb. Theory, Ser. B*, 53(1):1–4, 1991. doi:10.1016/0095-8956(91)90050-T.

[17] J. Kratochvíl and J. Matoušek. Intersection graphs of segments. *J. Comb. Theory, Ser. B*, 62(2):289–315, 1994. doi:10.1006/jctb.1994.1071.

[18] J. Matoušek. *Lectures on discrete geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer, 2002.

[19] J. Matoušek. Intersection graphs of segments and $\exists\mathbb{R}$. *arXiv*, 1406.2636, 2014.

[20] T. A. McKee and F. McMorris. *Topics in Intersection Graph Theory*. Society for Industrial and Applied Mathematics, 1999. doi:10.1137/1.9780898719802.

[21] I. Mustaţă, K. Nishikawa, A. Takaoka, S. Tayu, and S. Ueno. On orthogonal ray trees. *Discrete Applied Mathematics*, 201:201–212, 2016. doi:10.1016/j.dam.2015.07.034.

[22] W. Naji. Reconnaissance des graphes de cordes. *Discrete Mathematics*, 54(3):329 – 337, 1985. doi:10.1016/0012-365X(85)90117-7.

[23] A. Rok and B. Walczak. Outerstring graphs are χ-bounded. In *Proc. SoCG*, pages 136–143. ACM, 2014. doi:10.1145/2582112.2582115.

[24] M. Schaefer. Complexity of some geometric and topological problems. In *Proc. GD*, volume 5849 of *LNCS*, pages 334–344. Springer, 2009. doi:10.1007/978-3-642-11805-0_32.

[25] M. Schaefer, E. Sedgwick, and D. Štefankovič. Recognizing string graphs in NP. *J. Comput. Syst. Sci.*, 67(2):365–380, 2003. doi:10.1016/S0022-0000(03)00045-X.

[26] P. W. Shor. Stretchability of pseudolines is NP-hard. In *Applied Geometry And Discrete Mathematics*, volume 4 of *DIMACS Series in DMTCS*, pages 531–554. AMS, 1990.

[27] A. M. S. Shrestha, S. Tayu, and S. Ueno. On orthogonal ray graphs. *Discrete Applied Mathematics*, 158(15):1650–1659, 2010. doi:10.1016/j.dam.2010.06.002.

[28] F. W. Sinden. Topology of thin film RC circuits. *Bell System Technical Journal*, 45(9):1639–1662, 1966. doi:10.1002/j.1538-7305.1966.tb01713.x.

[29] J. A. Soto and C. Telha. Jump number of two-directional orthogonal ray graphs. In *Proc. IPCO*, volume 6655 of *LNCS*, pages 389–403. Springer, 2011. doi:10.1007/978-3-642-20807-2_31.

[30] W. Wessel and R. Pöschel. On circle graphs. In H. Sachs, editor, *Graphs, Hypergraphs and Applications*, pages 207–210. Teubner, 1985.