Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 17, no. 5, pp. 567–573 (2013)
DOI: 10.7155/jgaa.00304

An Approximate Restatement of the Four-Color Theorem

Atish Das Sarma[1] *Amita S. Gajewar*[2] *Richard J. Lipton*[3]
Danupon Nanongkai[4]

[1]eBay Research Labs, eBay Inc., CA, USA
[2]Microsoft Inc., CA, USA
[3]Georgia Institute of Technology, GA, USA
[4]University of Vienna, Vienna, Austria

Abstract

The celebrated Four-Color Theorem was first conjectured in the 1850's. Since then there had been many partial results. More than a century later, it was first proved by Appel and Haken [1] and then subsequently improved by Robertson et al. [8]. These proofs make extensive use of computer for various computations involved. In mathematical community, there continues to be an interest for a proof that is theoretical in nature. Our result provides an interesting restatement of the Four-Color Theorem that requires only approximate colorings. Tait proved in 1880 [11] that the Four-Color Theorem is equivalent to showing that two-edge connected, cubic, planar graphs have edge 3-colorings. Our main result is that this can be weakened to show that if there exists an *approximate edge 3-coloring* for these graphs, then the Four-Color Theorem is true.

Submitted:	Reviewed:	Revised:	Accepted:	Final:
February 2013	April 2013	June 2013	June 2013	July 2013
		Published: July 2013		

Article type: Concise paper
Communicated by: D. Wagner

Work done while authors were at Georgia Institute of Technology.
E-mail addresses: atish.dassarma@gmail.com (Atish Das Sarma) sgamita@gmail.com (Amita S. Gajewar) rjl@cc.gatech.edu (Richard J. Lipton) danupon@gmail.com (Danupon Nanongkai)

1 Introduction

We first state the Four-Color Theorem (4CT) and then provide several results related to an attempt in proving it.

Theorem 1 (Four-Color Theorem) *Every planar graph is face 4-colorable.*

One of the earliest results related to the 4CT is by Tait [11]:

Theorem 2 ([11]) *Let G be a two-edge connected, cubic, planar graph. G has an edge 3-coloring if and only if G has a face 4-coloring.*

In the proof of Theorem 2, Tait shows that if a graph G has an edge 3-coloring, then the face 4-coloring can be constructed in linear time. This fact will be used later on in our work.

In the hope of proving the 4CT, multiple restatements of the 4CT have been established. In [9], a survey of several equivalents forms of the 4CT is presented along with references to the original or related papers. Appel and Haken [1] were the first one to prove the 4CT. Their proof is based on an unavoidable set of configurations of every planar triangulation and relies on a computer to verify the various configurations. Robertson et al. [8] simplified the proof, cutting the number of configurations from 1500 to 633. Presently both of them are computer-aided, and hence impossible to humanly verify.

There continues to be considerable interest in finding a proof of the 4CT that does not rely on computer-aided search. In this spirit, our paper introduces new restatements of the 4CT. Before stating our results, we provide relevant definitions here. We say v is an *error-vertex* in an edge 3-coloring of a cubic, planar graph G if any two edges incident on v have the same color. And we define the number of *errors* in a coloring of G as the number of error-vertices. Further, we denote the number of vertices in G by n. Now, we state the contributions of our paper as the following two theorems. Note that we regard the 4CT as a conjecture (otherwise, the theorem statement is trivial).

Theorem 3 (Size Bound Theorem) *Every two-edge connected, cubic, planar graph G can be edge 3-colored with $o(n)$ errors $\implies$ 4CT.*

Theorem 4 (Diameter Bound Theorem) *For some constants d and $\epsilon < \frac{1}{2}$, every two-edge connected, cubic, planar graph G with $diam(G) > d$ can be edge 3-colored with $< 2^{\epsilon \cdot diam(G)}$ errors $\implies$ 4CT.*

It is easy to observe that cubic graphs have diameter at least $\log n$. So the statement of Theorem 4 is strictly weaker than Theorem 3 for graphs G with $\log n \leq \text{diam}(G) \leq \frac{1}{\epsilon} \log o(n)$. However, for graph with $\text{diam}(G) > \frac{1}{\epsilon} \log o(n)$, we do not need to prove anything if we are using Theorem 4 but would still need to prove $o(n)$ bound if we are using Theorem 3. Therefore, both these theorems provide different results.

We now compare our results with Tait's theorem. Brook's theorem [3] states that every cubic graph is edge colorable using three or four colors. Tait's approach needs every two-edge connected, cubic, and planar graph be edge colorable using just three colors. One way of viewing our result is that to prove

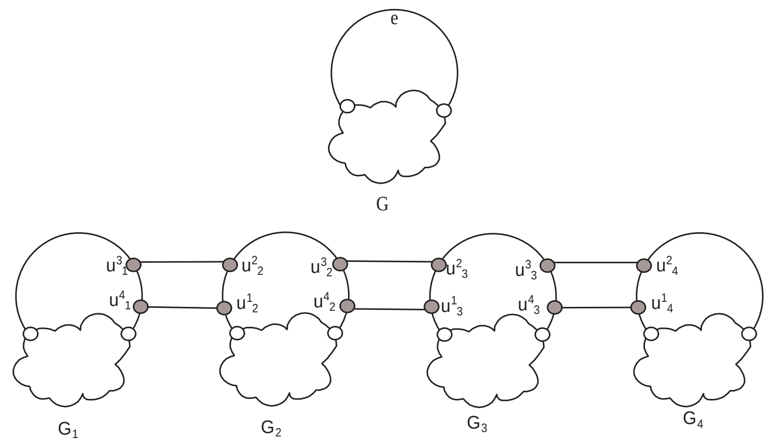

Figure 1: An example of constructing G' from four copies of G.

the 4CT, it is sufficient to find an algorithm to color these graphs using four colors such that the fourth color is used on $o(n)$ edges, where n is the number of vertices in the graph.

We conclude this section by mentioning the work related to approximate graph coloring. Since coloring 3-colorable graphs with three colors is NP-hard [6, 10], the problems of minimizing the number of colors [2, 5, 12] and maximizing the number of non-monochromatic edges [4, 7] are studied extensively. However, our approach differs from these problems in that we do not compare our solution to the optimal solution; rather we want to compare it to the number of vertices in one case and to the graph diameter in another.

In the following sections we provide the proofs of our Theorems 3 and 4. The main idea for both of these proofs is to construct an amplified graph with many copies of G and select a very small ϵ so that we can isolate at least one copy of G that has no errors. It is important to note that our proofs are existential.

2 Proof of Size Bound Theorem 3

The condition that every two-edge connected, cubic, planar graph G can be edge 3-colored with $o(n)$ errors is equivalent to the following statement.

For every ϵ, there exists n_0 such that every two-edge connected, cubic, planar graph with $n \geq n_0$ vertices is edge 3-colorable with at most ϵn errors. (1)

Assume (1) holds. We show that G is face 4-colorable.

Line construction: Select $\epsilon = \frac{1}{n+4}$. We construct a planar graph G' from n_0 copies of G as follows (Figure 1). For $i = 1, 2, ..., n_0$, let G_i be a copy of G with an outer edge $e \in E(G)$ subdivided two times for $i = 1$ and n_0, and four times for other values of i. Denote the new vertices in G_1 by u_1^3 and u_1^4, in G_{n_0} by $u_{n_0}^1$ and $u_{n_0}^2$, and in G_i, for other values of i, by $u_i^1, u_i^2, u_i^3, u_i^4$. For every $1 \leq i < n_0$, add an edge $u_i^3 u_{i+1}^2$ and $u_i^4 u_{i+1}^1$.

Lemma 1 *There is an edge 3-coloring of G' such that for some i, the subdivision of G_i, along with its incident edges, has no errors.*

Proof: By construction, G' is cubic, planar, and, since no bridge is introduced, it is two-edge connected. We now apply (1) to G'. Since $|V(G')| = n_0(n+4) - 4 \geq n_0$, there exists an edge 3-coloring C of G' such that there are at most $\epsilon|V(G')| = \frac{n_0(n+4)-4}{n+4} < n_0$ errors. Therefore, there exists $1 \leq i \leq n_0$ such that C colors G_i with no error vertices. G_i with its incident edges is the desired graph. □

Let H^* be the subgraph G_i guaranteed by Lemma 1. Let H be the union of H^* and the two or four incident edges.

Lemma 2 *H^* is face 4-colorable.*

Proof: Suppose that H has k incident edges $e_1, e_2, ..., e_k$ (where k is either 2 or 4). Let C be an edge 3-coloring of H. Let H_1 and H_2 be two copies of H but let H_2 be drawn in such a way that it is a mirror (or horizontal flip) of H. We construct a cubic, planar graph H' by connecting and merging together each pair of edges e_i in H_1 and H_2. Extend C to an edge 3-coloring of H' by coloring each edge in H' with the same color as its corresponding edge in H (See Figure 2). Note that by construction, H' is cubic, planar and two-edge connected. Therefore, by Tait's theorem 2, H' has a face 4-coloring C'. To obtain a face 4-coloring of H^* color every internal face the same way it is colored by C' and color the unbounded face the same color of unbounded face of H'. Since any pair of adjacent faces of H^* are also adjacent in H', it is a valid face 4-coloring of H^*. The crucial point is that the face originally in G adjacent to edge e is still adjacent to the *outer* face, as we had only subdivided the edge e, without drawing additional edges from the original vertices of e. □

Since H^* is a subdivision of G, we can obtain a face 4-coloring of G from the face 4-coloring of H^*. For an example, see Figure 3. This completes the proof of Theorem 3.

3 Proof of Diameter Bound Theorem 4

We use the tree construction in Figure 4 to obtain a graph G' formed by a binary tree along with h copies of G. For every copy of G, we shall subdivide an outer edge twice (for the leftmost and rightmost copy of G) or thrice and draw edges from the new vertices. The central edge will connect to a leaf of the tree, while the left and right vertices will connect to the left and right leaf copies of G in the tree.

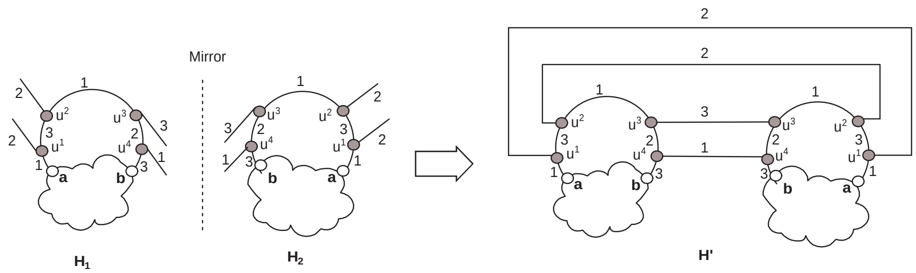

Figure 2: An example of extending an edge 3-coloring.

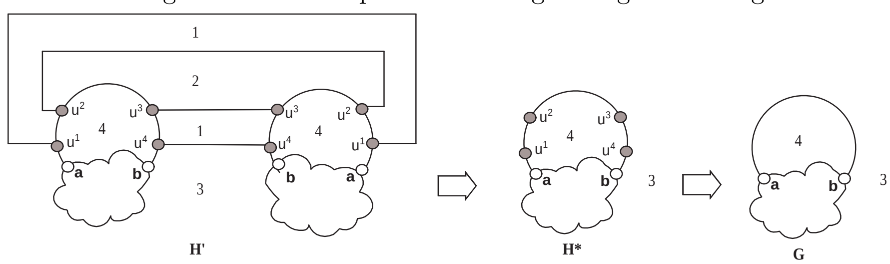

Figure 3: An example of a reaching a valid face 4-coloring of G from a valid edge 3-coloring of H'.

Suppose for all G with $\mathrm{diam}(G) > d$, we have errors $< 2^{\epsilon \cdot \mathrm{diam}(G)}$ for some $\epsilon < \frac{1}{2}$. We choose h such that $h = max\{2^d, 2^{\frac{2\epsilon n}{1-2\epsilon}}\}$. By construction of G', we have,

$$\begin{aligned} \mathrm{diam}(G') &\leq 2 \cdot \mathrm{diam}(G) + 2 \log h \\ &\leq 2n + 2 \log h \end{aligned}$$

To use the previous technique of isolating one copy of G with no edge-coloring errors, we require the following condition to hold true,

$$h \geq 2^{\epsilon \cdot \mathrm{diam}(G')}$$

which is implied by the following

$$h \geq 2^{\epsilon(2n+2\log h)}$$

Notice that by our choice of h, it follows that $\mathrm{diam}(G') > \log h \geq d$. Therefore, we have errors $< 2^{\epsilon(2n+2\log h)}$. So what we need is the following.

$$\begin{aligned} h &\geq 2^{\epsilon(2n+2\log h)} \\ \implies h &\geq \{2^n\}^{\frac{2\epsilon}{1-2\epsilon}} \end{aligned}$$

However, this condition is again true by our choice of h. Therefore, we can isolate at least one copy of G with no error vertices. Using the technique in

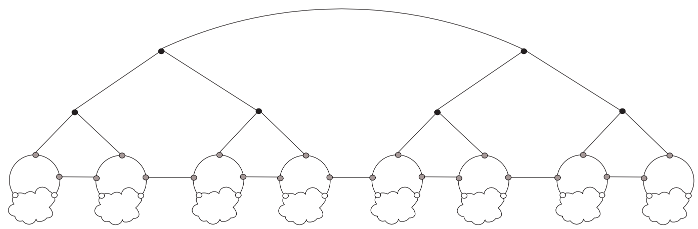

Figure 4: Tree construction

the proof of Theorem 3 (starting with the *mirror trick*), we can obtain a face 4-coloring of G. This completes the proof of Theorem 4.

4 Future Work

We conclude with an open question of whether there are simple proofs for the existence of, or algorithms for the computation of edge 3-colorings as required by our two main Theorems 3 and 4. Further, it might be interesting to improve the error bound in Theorem 4 to $o(2^{\mathrm{diam}(G)})$. We would then need to only consider graphs with diameter $\log n + O(1)$ and prove an error bound of $o(n)$, thereby generalizing both our theorems.

Acknowledgements

We would like to thank Nisheeth Vishnoi for his kind help with the preliminary ideas of using approximate colorings as an approach to the 4CT.

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